INTEGRABLES HAMILTONIAN SYSTEMS AND THE ISOSPECTRAL DEFORMATION METHOD

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ABSTRACT

Integrable hamiltonian systems are nonlinear ordinary differential equations described by a hamiltonian function and possessing sufficiently many independent constants of motion in involution. The regular compact level manifolds defined by the intersection of the constants of motion are diffeomorphic to a real torus on which the motion is quasi-periodic as a consequence of the following purely differential geometric fact : a compact and connected n-dimensional manifold on which there exist n vector fields which commute and are independent at every point is diffeomorphic to an n-dimensional real torus and each vector field will define a linear flow there. New examples of completely integrable hamiltonian systems, which have recently been discovered, are based on the Lax representation of the equations of motion. These systems can be realized as straight line motions on a Jacobi variety of a so-called spectral curve. We make a careful study of the connection with the concept of completely integrable systems and we apply the methods to several problems.

Keywords: Integrable hamiltonian systems, Lax representation, Jacobi variety.

1 INTEGRABLES HAMILTONIAN SYSTEMS AND SPECTRAL CURVE

Let M be an even-dimensional differentiable manifold. A symplectic structure (or symplectic form) on M is a closed non-degenerate differential 2-form ω defined everywhere on M. The non-degeneracy condition means that

$$\forall x \in M, \forall \xi \neq 0, \exists \eta : \omega(\xi, \eta) \neq 0, (\xi, \eta \in T_x M).$$

The pair (M, ω) is called a symplectic manifold.

Example 1.1 The cotangent bundle T^*M possesses in a natural way a symplectic structure. In a local coordinate $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, $2n = \dim M$, the form ω is given by $\omega = \sum_{k=1}^n dx_k \wedge dy_k$.

Example 1.2 Another important class of symplectic manifolds consists of the coadjoints orbits $\mathcal{O} \subset \mathcal{G}^*$, where $\mathcal{G}$ is the algebra of a Lie group $\mathcal{G}$ and $\mathcal{G}_\mu = \{Ad_g^* \mu : g \in \mathcal{G}\}$ is the orbit of $\mu \in \mathcal{G}^*$ under the coadjoint representation.

Proposition 1.1 Let $I : T_x^*M \longrightarrow T_xM,\ \omega_\xi^1 \longmapsto \xi$, be a map defined by $\omega_\xi^1(\eta) = \omega(\eta, \xi)$, $\forall \eta \in T_xM$. Then I is an isomorphism generated by the symplectic form ω.

Proof. Denote by I^{-1} the map

$$I^{-1} : T_xM \longrightarrow T_x^*M,\ \xi \longmapsto I^{-1}(\xi) \equiv \omega_\xi^1,$$

with $I^{-1}(\xi)(\eta) = \omega_\xi^1(\eta) = \omega(\eta, \xi)$, $\forall \eta \in T_xM$. The fact that the form ω is bilinear implies that

$$\begin{aligned} I^{-1}(\xi_1 + \xi_2)(\eta) &= \omega(\eta, \xi_1 + \xi_2), \\ &= \omega(\eta, \xi_1) + \omega(\eta, \xi_2), \\ &= I^{-1}(\xi_1)(\eta) + I^{-1}(\xi_2)(\eta),\ \forall \eta \in T_xM. \end{aligned}$$

Now, since $\dim T_xM = \dim T_x^*M$, to show that I^{-1} is bijective, it suffices to show that is injective. The form ω is non-degenerate, it follows that

$$KerI^{-1} = \{\xi \in T_xM : \omega(\eta, \xi) = 0,\ \forall \eta \in T_xM\} = \{0\}.$$

Hence I^{-1} is an isomorphism and consequently I is also an isomorphism (the inverse of an isomorphism is an isomorphism).□

As a consequence, the symplectic form ω induces a hamiltonian vector field $IdH : M \longrightarrow T_xM,\ x \longmapsto IdH(x)$, where $H : M \longrightarrow \mathbb{R}$, is a differentiable function (called hamiltonian). In others words, the differential system defined by

$$\dot{x}(t) = X_H(x(t)) = IdH(x),$$

is a hamiltonian vector field associated to the function H.

Proposition 1.2 The matrix that is associated to an hamiltonian system determine a symplectic structure.

Proof. Let $(x_1, \ldots, x_m)$ be a local coordinate system on M, $(m = \dim M)$. We have

$$\dot{x}(t) = \sum_{k=1}^{n} \frac{\partial H}{\partial x_k} I(dx_k) = \sum_{k=1}^{n} \frac{\partial H}{\partial x_k} \xi^k, \tag{1}$$

where $I(dx_k) = \xi^k \in T_xM$ is defined such that :

$$\forall \eta \in T_xM,\ \eta_k = dx_k(\eta) = \omega\left(\eta, \xi^k\right), \quad \text{(k-th component of}\eta).$$

Define $(\eta_1, \ldots, \eta_m)$ and $\left(\xi_1^k, \ldots, \xi_m^k\right)$ to be respectively the components of η and ξ^k, then

$$\begin{aligned} \eta_k &= \omega\left(\sum_{i=1}^{m} \eta_i \frac{\partial}{\partial x_i}, \sum_{j=1}^{m} \xi_j^k \frac{\partial}{\partial x_j}\right), \\ &= \sum_{i=1}^{m} \eta_i \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \xi_j^k, \\ &= (\eta_1, \ldots, \eta_m) J^{-1} \begin{pmatrix} \xi_1^k \\ \vdots \\ \xi_m^k \end{pmatrix}, \end{aligned}$$

where J^{-1} is the matrix defined by $J^{-1} \equiv \left(\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right)_{1\leq i,j\leq m}$. We shall show that this matrix is invertible. Indeed, it suffices to show that the matrix J^{-1} has maximal rank. Suppose this were not possible, i.e., we assume that $rank(J^{-1}) \neq m$. Hence $\sum_{i=1}^m a_i\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = 0$, $\forall 1 \leq j \leq m$, with a_i not all null and $\omega\left(\sum_{i=1}^m a_i\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = 0, \forall 1 \leq j \leq m$. In fact, since ω is non-degenerate, we have $\sum_{i=1}^m a_i\frac{\partial}{\partial x_i} = 0$. Moreover, since $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right)$ is a basis of T_xM, then $a_i = 0$, $\forall i$, contradiction. The matrix J^{-1} is invertible and we can search ξ^k such that :

$$J^{-1}\begin{pmatrix} \xi_1^k \\ \vdots \\ \xi_m^k \end{pmatrix} = \begin{pmatrix} 0 & \\ \vdots & \\ 0 & \\ 1 & \leftrightsquigarrow \text{k-th place} \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix}.$$

The matrix J^{-1} is invertible, which implies

$$\begin{pmatrix} \xi_1^k \\ \vdots \\ \xi_m^k \end{pmatrix} = J\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

from which ξ^k =(k-th column of J), i.e., $\xi_i^k = J_{ik}$, $1 \leq i \leq m$, and consequently $\xi^k = \sum_{i=1}^m J_{ik}\frac{\partial}{\partial x_i}$. It is easily verified that the matrix J is skew-symmetric. Indeed, since ω is symmetric i.e., $\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = -\omega\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}\right)$, it follows that J^{-1} is skew-symmetric. Then, $I = J.J^{-1} = (J^{-1})^\top.J^\top = -J^{-1}.J^\top$ and consequently $J^\top = -J$. From (1) we deduce that

$$\dot{x}(t) = \sum_{k=1}^m \frac{\partial H}{\partial x_k}\sum_{i=1}^m J_{ik}\frac{\partial}{\partial x_i} = \sum_{i=1}^m\left(\sum_{k=1}^m J_{ik}\frac{\partial H}{\partial x_k}\right)\frac{\partial}{\partial x_i}.$$

Writing $\dot{x}(t) = \sum_{i=1}^m \frac{dx_i(t)}{dt}\frac{\partial}{\partial x_i}$, it is seen that

$$\dot{x}_i\left(t\right) = \sum_{k=1}^m J_{ik}\frac{\partial H}{\partial x_k},\ 1 \leq i \leq j \leq m,$$

which can be written in more compact form

$$\dot{x}(t) = J\left(x\right)\frac{\partial H}{\partial x},$$

this is the hamiltonian vector field associated to the function H. □

We define a Poisson bracket (or Poisson structure) on the space $\mathcal{C}^{\infty}$ as

$$\{,\}:\mathcal{C}^{\infty}(M)\times\mathcal{C}^{\infty}(M)\longrightarrow\mathcal{C}^{\infty}(M),\ \ (F,G)\longmapsto\{F,G\},$$

where $\{F,G\}=d_uF(X_G)=X_GF(u)=\omega(X_G,X_F)$. This bracket is skew-symmetric $\{F,G\}=-\{G,F\}$, obeys the Leibniz rule $\{FG,H\}=F\{G,H\}+G\{F,H\}$, and satisfies the Jacobi identity

$$\{\{H,F\},G\}+\{\{F,G\},H\}+\{\{G,H\},F\}=0.$$

When this Poisson structure is non-degenerate, we obtain the symplectic structure discussed above.

Consider now $M=\mathbb{R}^n\times\mathbb{R}^n$ and let $p\in M$. By Darboux's theorem (Arnold 1978), there exists a local coordinate system$(x_1,\ldots,x_n,y_1,\ldots,y_n)$ in a neighbourhood of p such that

$$\{H,F\}=\sum_{i=1}^{n}\left(\frac{\partial H}{\partial x_i}\frac{\partial F}{\partial y_i}-\frac{\partial H}{\partial y_i}\frac{\partial F}{\partial x_i}\right).$$

Then $X_H=\sum_{i=1}^{n}\left(\frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i}-\frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i}\right)$, and $X_HF=\{H,F\}$, $\forall F\in\mathcal{C}^{\infty}(M)$. A nonconstant function F is called an integral (first integral or constant of motion) of X_F, if $X_HF=0$. In particular, H is integral. Two functions F and G are said to be in involution or to commute, if $\{F,G\}=0$. The hamiltonian systems form a Lie algebra.

We now give the following definition of the Poisson bracket :

$$\{F,G\}=\left\langle\frac{\partial F}{\partial x},J\frac{\partial G}{\partial x}\right\rangle=\sum_{i,j}J_{ij}\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j}.$$

Proposition 1.3 If

$$\sum_{k=1}^{2n}\left(J_{kj}\frac{\partial J_{li}}{\partial x_k}+J_{ki}\frac{\partial J_{jl}}{\partial x_k}+J_{kl}\frac{\partial J_{ij}}{\partial x_k}\right)=0,\ \forall 1\leq i,j,l\leq 2n,$$

then J satisfies the Jacobi identity.

Proof. Consider the the Jacobi identity

$$\{\{H,F\},G\}+\{\{F,G\},H\}+\{\{G,H\},F\}=0,$$

where $\{H,F\}=\left\langle\frac{\partial H}{\partial x},J\frac{\partial F}{\partial x}\right\rangle=\sum_{i,j}J_{ij}\frac{\partial H}{\partial x_i}\frac{\partial F}{\partial x_j}$. We have

$$\begin{aligned}\{\{H,F\},G\} &= \left\langle\frac{\partial\{H,F\}}{\partial x},J\frac{\partial G}{\partial x}\right\rangle,\\ &= \sum_{k,l}J_{kl}\frac{\partial\{H,F\}}{\partial x_k}\frac{\partial G}{\partial x_l},\\ &= \sum_{k,l}\sum_{i,j}J_{kl}\frac{\partial J_{ij}}{\partial x_k}\frac{\partial H}{\partial x_i}\frac{\partial F}{\partial x_j}\frac{\partial G}{\partial x_l}+\sum_{k,l}\sum_{i,j}J_{kl}J_{ij}\frac{\partial^2 H}{\partial x_k\partial x_i}\frac{\partial F}{\partial x_j}\frac{\partial G}{\partial x_l}\\ &\quad+\sum_{k,l}\sum_{i,j}J_{kl}J_{ij}\frac{\partial H}{\partial x_i}\frac{\partial^2 F}{\partial x_k\partial x_j}\frac{\partial G}{\partial x_l}.\end{aligned}$$

By symmetry, we have immediately $\{\{F,G\},H\}$ and $\{\{G,H\},F\}$. Then

$$\begin{aligned}
\{\{H,F\},G\}+\{\{F,G\},H\} \quad + \quad & \{\{G,H\},F\} \\
= \quad & \sum_{k,l}\sum_{i,j} J_{kl}\frac{\partial J_{ij}}{\partial x_k}\frac{\partial H}{\partial x_i}\frac{\partial F}{\partial x_j}\frac{\partial G}{\partial x_l} \\
& +\sum_{k,l}\sum_{i,j} J_{kl}J_{ij}\frac{\partial^2 H}{\partial x_k \partial x_i}\frac{\partial F}{\partial x_j}\frac{\partial G}{\partial x_l} \qquad (2) \\
& +\sum_{k,l}\sum_{i,j} J_{kl}J_{ij}\frac{\partial H}{\partial x_i}\frac{\partial^2 F}{\partial x_k \partial x_j}\frac{\partial G}{\partial x_l} \qquad (3) \\
& +\sum_{k,l}\sum_{i,j} J_{kl}\frac{\partial J_{ij}}{\partial x_k}\frac{\partial G}{\partial x_i}\frac{\partial H}{\partial x_j}\frac{\partial F}{\partial x_l} \\
& +\sum_{k,l}\sum_{i,j} J_{kl}J_{ij}\frac{\partial^2 G}{\partial x_k \partial x_i}\frac{\partial H}{\partial x_j}\frac{\partial F}{\partial x_l} \qquad (4) \\
& +\sum_{k,l}\sum_{i,j} J_{kl}J_{ij}\frac{\partial G}{\partial x_i}\frac{\partial^2 H}{\partial x_k \partial x_j}\frac{\partial F}{\partial x_l} \qquad (5) \\
& +\sum_{k,l}\sum_{i,j} J_{kl}\frac{\partial J_{ij}}{\partial x_k}\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j}\frac{\partial H}{\partial x_l} \\
& +\sum_{k,l}\sum_{i,j} J_{kl}J_{ij}\frac{\partial^2 F}{\partial x_k \partial x_i}\frac{\partial G}{\partial x_j}\frac{\partial H}{\partial x_l} \qquad (6) \\
& +\sum_{k,l}\sum_{i,j} J_{kl}J_{ij}\frac{\partial F}{\partial x_i}\frac{\partial^2 G}{\partial x_k \partial x_j}\frac{\partial H}{\partial x_l}. \qquad (7)
\end{aligned}$$

Notice that the indices i, j, k and l play a symmetric roll. Applying in the term (5) the permutation $i \leftarrow l$, $j \leftarrow k$, $k \leftarrow i$, $l \leftarrow j$, and add the term (2), with the understanding that $J_{lk} = -J_{kl}$, we get

$$\sum_{k,l}\sum_{i,j}\left(J_{ij}J_{lk}+J_{kl}J_{ij}\right)\frac{\partial G}{\partial x_l}\frac{\partial^2 H}{\partial x_i \partial x_k}\frac{\partial F}{\partial x_j}=0,$$

as a consequence of the Schwartz's lemma. Again applying in the term (6) the permutation $i \leftarrow k$, $j \leftarrow l$, $k \leftarrow j$, $l \leftarrow i$, and add the term (3), yields

$$\sum_{k,l}\sum_{i,j}\left(J_{ji}J_{kl}+J_{kl}J_{ij}\right)\frac{\partial^2 F}{\partial x_j \partial x_k}\frac{\partial G}{\partial x_l}\frac{\partial H}{\partial x_i}=0.$$

By the same argument as above, applying in the term (7) the permutation $i \leftarrow l$, $j \leftarrow k$, $k \leftarrow i$, $l \leftarrow j$, and add the term (4), we obtain

$$\sum_{k,l}\sum_{i,j}\left(J_{ij}J_{lk}+J_{kl}J_{ij}\right)\frac{\partial F}{\partial x_l}\frac{\partial^2 G}{\partial x_i \partial x_k}\frac{\partial H}{\partial x_j}=0,$$

and thus

$$\begin{aligned}\{\{H,F\},G\}+\{\{F,G\},H\} &+ \{\{G,H\},F\}\\ &= \sum_{k,l}\sum_{i,j} J_{kl}\frac{\partial J_{ij}}{\partial x_k}\frac{\partial H}{\partial x_i}\frac{\partial F}{\partial x_j}\frac{\partial G}{\partial x_l} \quad (8)\\ &\quad +\sum_{k,l}\sum_{i,j} J_{kl}\frac{\partial J_{ij}}{\partial x_k}\frac{\partial G}{\partial x_i}\frac{\partial H}{\partial x_j}\frac{\partial F}{\partial x_l} \quad (9)\\ &\quad +\sum_{k,l}\sum_{i,j} J_{kl}\frac{\partial J_{ij}}{\partial x_k}\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j}\frac{\partial H}{\partial x_l}.\end{aligned}$$

Under permuting the indices $i \leftarrow l$, $j \leftarrow i$, $k \leftarrow k$, $l \leftarrow j$, for (8) and $i \leftarrow j$, $j \leftarrow l$, $k \leftarrow k$, $l \leftarrow i$, for (9) , we obtain the following :

$$\begin{aligned}\{\{H,F\},G\} &+ \{\{F,G\},H\}+\{\{G,H\},F\}\\ &= \sum_{i,j,l}\left[\sum_k\left(J_{kj}\frac{\partial J_{li}}{\partial x_k}+J_{ki}\frac{\partial J_{jl}}{\partial x_k}+J_{kl}\frac{\partial J_{ij}}{\partial x_k}\right)\right]\frac{\partial H}{\partial x_l}\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j}.\end{aligned}$$

Since the Jacobi identity must be identically zero, we have

$$\sum_{k=1}^{2n}\left(J_{kj}\frac{\partial J_{li}}{\partial x_k}+J_{ki}\frac{\partial J_{jl}}{\partial x_k}+J_{kl}\frac{\partial J_{ij}}{\partial x_k}\right)=0,\ \forall 1\leq i,j,l\leq 2n,$$

ending the proof of proposition.□

Consequently, we have a complete characterization of hamiltonian vector field

$$\dot{x}(t)=X_H(x(t))=J\frac{\partial H}{\partial x},\ x\in M, \qquad (10)$$

where $H : M \longrightarrow \mathbb{R}$, is a differentiable function (the hamiltonian) and $J = J(x)$ is a skew-symmetric matrix, possibly depending on $x \in M$, for which the corresponding Poisson bracket satisfies the Jacobi identity :

$$\{\{H,F\},G\}+\{\{F,G\},H\}+\{\{G,H\},F\}=0,$$

with $\{H,F\} = \left\langle \frac{\partial H}{\partial x}, J\frac{\partial F}{\partial x}\right\rangle = \sum_{i,j} J_{ij}\frac{\partial H}{\partial x_i}\frac{\partial F}{\partial x_j}$, the Poisson bracket.

Example 1.3 An important special case is when $J = \begin{pmatrix} O & -I \\ I & O\end{pmatrix}$, where I is the $n\times n$ identity matrix. The condition on J (see proposition 1.3) is trivially satisfied. Indeed, here the matrix J do not depend on the variable x and we have

$$\{H,F\}=\sum_{i=1}^{2n}\frac{\partial H}{\partial x_i}\sum_{j=1}^{2n}J_{ij}\frac{\partial F}{\partial x_j}=\sum_{i=1}^{n}\left(\frac{\partial H}{\partial x_{n+i}}\frac{\partial F}{\partial x_i}-\frac{\partial H}{\partial x_i}\frac{\partial F}{\partial x_{n+i}}\right).$$

Moreover, equations (10) are transformed into

$$\dot{q}_1=\frac{\partial H}{\partial p_1},\ldots,\dot{q}_n=\frac{\partial H}{\partial p_n},\ \dot{p}_1=-\frac{\partial H}{\partial q_1},\ldots,\dot{p}_n=-\frac{\partial H}{\partial q_n},$$

où $q_1 = x_1, \ldots, q_n = x_n, p_1 = x_{n+1}, \ldots, p_n = x_{2n}$. These are exactly the well known differential equations of classical mechanics in canonical form.

It is a fundamental and important problem to investigate the integrability of hamiltonian systems. Recently there has been much effort given for finding integrable hamiltonian systems, not only because they have been on the subject of powerful and beautiful theories of mathematics, but also because the concepts of integrability have been applied to an increasing number of applied sciences. The so-called Arnold-Liouville theorem play a crucial role in the study of such systems; the regular compact level manifolds defined by the intersection of the constants of motion are diffeomorphic to a real torus on which the motion is quasi-periodic as a consequence of the following purely differential geometric fact : a compact and connected n-dimensional manifold on which there exist n vector fields which commute and are independent at every point is diffeomorphic to an n-dimensional real torus and each vector field will define a linear flow there.

Proposition 1.4 (Arnold-Liouville theorem) (Arnold 1978; Lesfari 2003): Let $H_1 = H, H_2, ...,$ H_n, be n first integrals on a $2n$-dimensional symplectic manifold that are functionally independent (i.e., $dH_1 \wedge ... \wedge dH_n \neq 0$), and pairwise in involution. For generic $c = (c_1, ..., c_n)$ the level set

$$M_c = \bigcap_{i=1}^{n} \{x \in M : H_i(x) = c_i,\ c_i \in \mathbb{R}\},$$

will be an n-manifold. If M_c is compact and connected, it is diffeomorphic to an n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and the solutions of the system (10) are then straight-line motions on $\mathbb{T}^n$. If M_c is not compact but the flow of each of the vector fields X_{H_k} is complete on M_c, then M_c is diffeomorphic to a cylinder $\mathbb{R}^k \times \mathbb{T}^{n-k}$ under which the vector fields X_{H_k} are mapped to linear vector fields.

As a consequence, we obtain the concept of complete integrability of a hamiltonian system. For the sake of clarity, we shall distinguish two cases :
a) Case 1 : $dét\ J \neq 0$. The rank of the matrix J is even, $m = 2n$. A hamiltonian system (10) is completely integrable or Liouville-integrable if there exist n firsts integrals $H_1 = H, H_2, \ldots, H_n$ in involution, i.e., $\{H_k, H_l\} = 0,\ 1 \leq k, l \leq n$, with linearly independent gradients, i.e., $dH_1 \wedge ... \wedge dH_n \neq 0$. For generic $c = (c_1, ..., c_n)$ the level set

$$M_c = \bigcap_{i=1}^{n} \{x \in M : H_i(x) = c_i,\ c_i \in \mathbb{R}\},$$

will be an n-manifold. By the Arnold-Liouville theorem (proposition 1.4), if M_c is compact and connected, it is diffeomorphic to an n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and each vector field will define a linear flow there. In some open neighbourhood of the torus there are coordinates $s_1, \ldots, s_n, \varphi_1, \ldots, \varphi_n$ in which ω takes the form $\omega = \sum_{k=1}^{n} ds_k \wedge d\varphi_k$. Here the functions s_k (called action-variables) give coordinates in the direction transverse to the torus and can be expressed functionally in terms of the firsts integrals H_k. The functions φ_k (called angle-variables) give standard angular coordinates on the torus, and every vector field X_{H_k} can be written in the form $\dot{\varphi}_k = h_k(s_1, \ldots, s_n)$, that is, its integral trajectories define a conditionally-periodic motion on the torus. In a neighbourhood of the torus the hamiltonian vector field X_{H_k} take the following form $\dot{s}_k = 0, \dot{\varphi}_k = h_k(s_1, \ldots, s_n)$, and can be solved by quadratures.
b) Case 2 : $dét\ J = 0$. We reduce the problem to $m = 2n + k$ and we look for k Casimir functions (or trivial invariants) $H_{n+1}, ..., H_{n+k}$, leading to identically zero hamiltonian vector fields

$J\frac{\partial H_{n+i}}{\partial x}=0,\ 1\leq i\leq k$. In other words, the system is hamiltonian on a generic symplectic manifold

$$\bigcap_{i=n+1}^{n+k}\{x\in\mathbb{R}^m : H_i(x)=c_i\},$$

of dimension $m-k=2n$. If for most values of $c_i\in\mathbb{R}$, the invariant manifolds

$$\bigcap_{i=1}^{n+k}\{x\in\mathbb{R}^m : H_i(x)=c_i\},$$

are compact and connected, then they are n-dimensional tori $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$ by the Arnold-Liouville theorem and the hamiltonian flow is linear in angular coordinates of the torus.

A Lax equation is given by a differential equation of the form

$$\dot{A}(t)=[A(t),B(t)] \text{ or } [B(t),A(t)], \qquad (11)$$

where

$$A(t)=\sum_{k=1}^{N}A_k(t)h^k,\ B(t)=\sum_{k=1}^{N}B_k(t)h^k,$$

are functions depending on a parameter h (spectral parameter) whose coefficients A_k and B_k are matrices in Lie algebras. The pair (A,B) is called Lax pair. This equation established a link between the Lie group theoretical and the algebraic geometric approaches to complete integrability. The solution to (11) has the form $A(t)=g(t)A(0)g(t)^{-1}$, where $g(t)$ is a matrix defined as $\dot{g}(t)=-A(t)g(t)$. We form the polynomial

$$P(h,z)=\det(A-zI),$$

where z is another variable and I the $n\times n$ identity matrix. We define the curve (spectral curve) $\mathcal{C}$, to be the normalization of the complete algebraic curve whose affine equation is

$$P(h,z)=0. \qquad (12)$$

Proposition 1.5 The polynomial $P(h,z)$ is independent of t. Moreover, the functions $tr(A^n)$ are first integrals for (11).

Proof. Let us call $L\equiv A-zI$. Observe that

$$\dot{P}=\det L.tr\left(L^{-1}\dot{L}\right)=\det L.tr\left(L^{-1}BL-B\right)=0,$$

since $trL^{-1}BL=trB$. On the other hand

$$\begin{aligned}\dot{A}^n &= \dot{A}A^{n-1}+A\dot{A}A^{n-2}+\cdots+A^{n-1}\dot{A},\\ &= [A,B]A^{n-1}+A[A,B]A^{n-2}+\cdots+A^{n-1}[A,B],\\ &= (AB-BA)A^{n-1}+\cdots+A^{n-1}(AB-BA),\\ &= ABA^{n-1}-BA^n+\cdots+A^nB-A^{n-1}BA,\\ &= A\left(BA^{n-1}\right)-\left(BA^{n-1}\right)A+\cdots+A\left(A^{n-1}B\right)-\left(A^{n-1}B\right)A.\end{aligned}$$

Since $tr\,(X+Y) = trX + trY, trXY = trYX,\ X, Y \in \mathcal{M}_n\,(\mathbb{C})$, we obtain

$$\frac{d}{dt} tr\,(A_h^n) = tr \frac{d}{dt}\,(A_h^n) = 0,$$

and consequently $tr\,(A^n)$ are first integrals of motion.□

We have shown that a hamiltonian flow of the type (11) preserves the spectrum of A and therefore its characteristic polynomial. The curve $\mathcal{C} : P(z,h) = \det\,(A(h) - zI) = 0$, is time independent, i.e., its coefficients $tr\,(A^n)$ are integrals of the motion (equivalently, $A(t)$ undergoes an isospectral deformation. Some hamiltonian flows on Kostant-Kirillov coadjoint orbits in subalgebras of infinite dimensional Lie algebras (Kac-Moody Lie algebras) yield large classes of extended Lax pairs (11). A general statement leading to such situations is given by the Adler-Kostant-Symes theorem. Using the van Moerbeke-Mumford linearization method, Adler and van Moerbeke (Adler and van Moerbeke 1980) showed that the linearized flow could be realized on the jacobian variety $Jac(\mathcal{C})$ (or some sub-abelian variety of it) of the algebraic curve (spectral curve) $\mathcal{C}$ associated to (11). We then construct an algebraic map from the complex invariant manifolds of these hamiltonian systems to the jacobian variety $Jac(\mathcal{C})$ of the curve $\mathcal{C}$. Therefore all the complex flows generated by the constants of the motion are straight line motions on these jacobian varieties i.e. the linearizing equations are given by

$$\int_{s_1(0)}^{s_1(t)} \omega_k + \int_{s_2(0)}^{s_2(t)} \omega_k + \cdots + \int_{s_g(0)}^{s_g(t)} \omega_k = c_k t\ ,\ 0 \le k \le g,$$

where $\omega_1, \ldots, \omega_g$ span the g-dimensional space of holomorphic differentials on the curve $\mathcal{C}$ of genus g. In an unifying approach, Griffiths (Griffiths 1985) has found necessary and sufficient conditions on B for the Lax flow (11) to be linearizable on the jacobi variety of its spectral curve, without reference to Kac-Moody Lie algebras.

2 APPLICATIONS

In this section, we discuss a number of integrable hamiltonian systems.

2.1 The Euler Rigid Body Motion

It express the free motion of a rigid body around a fixed point. Let $M = (m_1, m_2, m_3)$ be the angular momentum, $\Omega = (m_1/I_1, m_2/I_2, m_3/I_3)$ the angular velocity and I_1, I_2 et I_3, the principal moments of inertia about the principal axes of inertia. Then the motion of the body is governed by

$$\dot{M} = M \wedge \Omega. \tag{13}$$

If one identifies vectors in $\mathbb{R}^3$ with skew-symmetric matrices by the rule

$$a = (a_1, a_2, a_3)\,, \quad A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix},$$

then $a \wedge b \longmapsto [A, B] = AB - BA$. Using this isomorphism between $(\mathbb{R}^3, \wedge)$ and $(so(3), [,])$, we write (13) as $\dot{M} = [M, \Omega]$, where

$$M = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \in so\,(3)\,, \quad \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in so\,(3)\,,$$

Now $M = I\Omega$, this implies that

$$\dot{M} = [M, \Lambda M]\,, \tag{14}$$

where

$$\Lambda M = \begin{pmatrix} 0 & -\lambda_3 m_3 & \lambda_2 m_2 \\ \lambda_3 m_3 & 0 & -\lambda_1 m_1 \\ -\lambda_2 m_2 & \lambda_1 m_1 & 0 \end{pmatrix} \in so\,(3)\,,$$

with $\lambda_i \equiv I_i^{-1}$. Equation (14) is explicitly given by

$$\begin{aligned} \dot{m}_1 &= (\lambda_3 - \lambda_2)\, m_2 m_3, \\ \dot{m}_2 &= (\lambda_1 - \lambda_3)\, m_1 m_3, \\ \dot{m}_3 &= (\lambda_2 - \lambda_1)\, m_1 m_2, \end{aligned} \tag{15}$$

and can be written as a hamiltonian vector field

$$\dot{x} = J\frac{\partial H}{\partial x},\ x = (m_1, m_2, m_3)^\intercal\,,$$

with the hamiltonian $H = \frac{1}{2}\left(\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2\right)$, and

$$J = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \in so\,(3)\,.$$

We have $\det J = 0$, so $m = 2n + k$ and $m - k = rk\ J$. Here $m = 3$ and $rk\ J = 2$, then $n = k = 1$. The system (15) has beside the energy $H_1 = H$, a trivial invariant H_2, i.e., such that: $J\frac{\partial H_2}{\partial x} = 0$, or

$$\begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H_2}{\partial m_1} \\ \frac{\partial H_2}{\partial m_2} \\ \frac{\partial H_2}{\partial m_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

implying $\frac{\partial H_2}{\partial m_1} = m_1,\ \frac{\partial H_2}{\partial m_2} = m_2,\ \frac{\partial H_2}{\partial m_3} = m_3$, and consequently

$$H_2 = \frac{1}{2}\left(m_1^2 + m_2^2 + m_3^2\right).$$

The system evolves on the intersection of the sphere $H_1 = c_1$ and the ellipsoid $H_2 = c_2$. In $\mathbb{R}^3$, this intersection will be isomorphic to two circles $\left(\text{with } \frac{c_2}{\lambda_3} < c_1 < \frac{c_2}{\lambda_1}\right)$. We shall show that the problem can be integrated in terms of elliptic functions, as Euler discovered using his then newly invented theory of elliptic integrals. Observe that the first equation of (15) reads

$$\frac{dm_1}{m_2 m_3} = (\lambda_3 - \lambda_2)\, dt, \tag{16}$$

where m_1, m_2 and m_3 are related by

$$\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 = c_1, \quad m_1^2 + m_2^2 + m_3^2 = c_2.$$

Therefore, if $\lambda_2 \neq \lambda_3$, we have

$$m_2 = \pm\sqrt{\frac{c_2\lambda_3 - c_1 + (\lambda_1 - \lambda_3)\, m_1^2}{\lambda_3 - \lambda_2}}, \quad m_3 = \pm\sqrt{\frac{c_1 - c_2\lambda_2 + (\lambda_2 - \lambda_1)\, m_1^2}{\lambda_3 - \lambda_2}}.$$

Substituting these expressions into (16), we find after integration that the system (15) amounts to an elliptic integral

$$\int_{m_1(0)}^{m_1(t)} \frac{dm}{\sqrt{(m^2 + a)\,(m^2 + b)}} = ct,$$

with respect to the elliptic curve

$$\mathcal{C} : w^2 = \left(z^2 + a\right)\left(z^2 + b\right), \tag{17}$$

with $a = \frac{c_2\lambda_3 - c_1}{\lambda_1 - \lambda_3}$, $b = \frac{c_1 - c_2\lambda_2}{\lambda_2 - \lambda_1}$, $c = \sqrt{(\lambda_1 - \lambda_3)\,(\lambda_2 - \lambda_1)}$. Then the functions $m_i(t)$ can be expressed in terms of theta-functions of t, according to the classical inversion of abelian integrals.

We shall use the Lax representation of the equations of motion to show that the linearized Euler flow can be realized on an elliptic curve isomorphic to the original elliptic curve (17). The solution to (14) has the form

$$M(t) = O(t)\, M(t)\, M^{\top}(t),$$

where $O(t)$ is one parameter sub-group of $SO(3)$. So the hamiltonian flow (14) preserves the spectrum of X and therefore its characteristic polynomial $\det(M - zI) = -z(z^2 + m_1^2 + m_2^2 + m_3^2)$. Unfortunately, the spectrum of a 3×3 skew- symmetric matrix provides only one piece of information; the conservation of energy does not appear as part of the spectral information. Therefore one is let to considering another formulation. The basic observation, due to Manakov (Manakov 1976), is that equation (14) is equivalent to the Lax equation

$$\dot{A} = [A, B],$$

where $A = M + \alpha h, \quad B = \Lambda M + \beta h,$ with a formal indeterminate h and

$$\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \quad \beta = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix},$$

$$\lambda_1 = \frac{\beta_3 - \beta_2}{\alpha_3 - \alpha_2}, \quad \lambda_2 = \frac{\beta_1 - \beta_3}{\alpha_1 - \alpha_3}, \quad \lambda_3 = \frac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1},$$

and all α_i distinct. The characteristic polynomial of A is

$$\begin{aligned} P(h, z) &= \det(A - zI), \\ &= \det(M + \alpha h - zI), \\ &= \prod_{j=1}^{3} (\alpha_j h - z) + \left(\sum_{j=1}^{3} \alpha_j m_j^2\right) h - \left(\sum_{j=1}^{3} m_j^2\right) z. \end{aligned} \tag{18}$$

The spectrum of the matrix $A = M + \alpha h$ as a function of $h \in \mathbb{C}$ is time independent and is given by the zeroes of the polynomial $P(h, z)$, thus defining an algebraic curve (spectral curve). Letting $w = h/z$, we obtain the following elliptic curve

$$z^2 \prod_{j=1}^{3} (\alpha_j w - 1) + 2H_1 w - 2H_2 = 0,$$

which is shown to be isomorphic to the original elliptic curve. Finally, we have the

Proposition 2.1 The Euler rigid body motion is a completely integrable system and the linearized flow can be realized on an elliptic curve.

2.2 The Manakov Geodesic Flow on the Group $SO(4)$

Consider the group $SO(4)$ and its Lie algebra $so(4)$ paired with itself, via the customary inner product $\langle X, Y \rangle = -\frac{1}{2}\, tr\ (X.Y)$, where

$$X = \begin{pmatrix} 0 & -x_3 & x_2 & -x_4 \\ x_3 & 0 & -x_1 & -x_5 \\ -x_2 & x_1 & 0 & -x_6 \\ x_4 & x_5 & x_6 & 0 \end{pmatrix} \in so(4).$$

A left invariant metric on $SO(4)$ is defined by a non-singular symmetric linear map $\Lambda : so(4) \longrightarrow so(4),\ X \longmapsto \Lambda.X$, and by the following inner product; given two vectors gX and gY in the tangent space $SO(4)$ at the point $g \in SO(4)$, $\langle gX, gY \rangle = \langle X, \Lambda^{-1}.Y \rangle$. Then the geodesic flow for this metric takes the following commutator form (Euler-Arnold equations) :

$$\dot{X} = [X, \Lambda.X], \tag{19}$$

where

$$\Lambda.X = \begin{pmatrix} 0 & -\lambda_3 x_3 & \lambda_2 x_2 & -\lambda_4 x_4 \\ \lambda_3 x_3 & 0 & -\lambda_1 x_1 & -\lambda_5 x_5 \\ -\lambda_2 x_2 & \lambda_1 x_1 & 0 & -\lambda_6 x_6 \\ \lambda_4 x_4 & \lambda_5 x_5 & \lambda_6 x_6 & 0 \end{pmatrix} \in so(4).$$

In view of the isomorphism between $(\mathbb{R}^6, \wedge)$, and $(so(4), [,])$ we write the system (18) as

$$\begin{aligned} \dot{x}_1 &= (\lambda_3 - \lambda_2) x_2 x_3 + (\lambda_6 - \lambda_5) x_5 x_6, \\ \dot{x}_2 &= (\lambda_1 - \lambda_3) x_1 x_3 + (\lambda_4 - \lambda_4) x_4 x_6, \\ \dot{x}_3 &= (\lambda_2 - \lambda_1) x_1 x_2 + (\lambda_5 - \lambda_4) x_4 x_5, \\ \dot{x}_4 &= (\lambda_3 - \lambda_5) x_3 x_5 + (\lambda_6 - \lambda_2) x_2 x_6, \\ \dot{x}_5 &= (\lambda_4 - \lambda_3) x_3 x_4 + (\lambda_1 - \lambda_6) x_1 x_6, \\ \dot{x}_6 &= (\lambda_2 - \lambda_4) x_2 x_4 + (\lambda_5 - \lambda_1) x_1 x_5. \end{aligned}$$

These equations can be written as a hamiltonian vector field

$$\dot{x}(t) = J \frac{\partial H}{\partial x},\ x \in \mathbb{R}^6, \tag{20}$$

with

$$H=\frac{1}{2}\langle X,\Lambda X\rangle=\frac{1}{2}\left(\lambda_1x_1^2+\lambda_2x_2^2+\cdots+\lambda_6x_6^2\right),$$

the hamiltonian and

$$J=\begin{pmatrix} 0 & -x_3 & x_2 & 0 & -x_6 & x_5\\ x_3 & 0 & -x_1 & x_6 & 0 & -x_4\\ -x_2 & x_1 & 0 & -x_5 & x_4 & 0\\ 0 & -x_6 & x_5 & 0 & -x_3 & x_2\\ x_6 & 0 & -x_4 & x_3 & 0 & -x_1\\ -x_5 & x_4 & 0 & -x_2 & x_1 & 0 \end{pmatrix}\in so(6).$$

We have $\det J=0$, so $m=2n+k$ and $m-k=rk\ J$. Here $m=6$ and $rg\ J=4$, then $n=k=2$. The system (20) has beside the energy $H_1=H$, two trivial constants of motion :

$$\begin{aligned} H_2 &= \frac{1}{2}\left(x_1^2+x_2^2+\cdots+x_6^2\right),\\ H_3 &= x_1x_4+x_2x_5+x_3x_6. \end{aligned}$$

Recall that H_2 and H_3 are called trivial invariants (or Casimir functions) because $J\frac{\partial H_2}{\partial x}=J\frac{\partial H_3}{\partial x}=0$. In order that the hamiltonian system (20) be completely integrable, it is suffices to have one more integral, which we take of the form

$$H_4=\frac{1}{2}\left(\mu_1x_1^2+\mu_2x_2^2+\cdots+\mu_6x_6^2\right).$$

The four invariants must be functionally independent and in involution, so in particular

$$\{H_4,H_3\}=\left\langle\frac{\partial H_4}{\partial x},J\frac{\partial H_3}{\partial x}\right\rangle=0,$$

i.e.,

$$\begin{aligned} &((\lambda_3-\lambda_2)\mu_1+(\lambda_1-\lambda_3)\mu_2+(\lambda_2-\lambda_1)\mu_3)x_1x_2x_3\\ &+((\lambda_6-\lambda_5)\mu_1+(\lambda_1-\lambda_6)\mu_5+(\lambda_5-\lambda_1)\mu_6)x_1x_5x_6\\ &+((\lambda_4-\lambda_6)\mu_2+(\lambda_6-\lambda_2)\mu_4+(\lambda_2-\lambda_4)\mu_6)x_2x_4x_6\\ &+((\lambda_5-\lambda_4)\mu_3+(\lambda_3-\lambda_5)\mu_4+(\lambda_4-\lambda_3)\mu_5)x_3x_4x_5=0. \end{aligned}$$

Then

$$\begin{aligned} &(\lambda_3-\lambda_2)\mu_1+(\lambda_1-\lambda_3)\mu_2+(\lambda_2-\lambda_1)\mu_3=0,\\ &(\lambda_6-\lambda_5)\mu_1+(\lambda_1-\lambda_6)\mu_5+(\lambda_5-\lambda_1)\mu_6=0,\\ &(\lambda_4-\lambda_6)\mu_2+(\lambda_6-\lambda_2)\mu_4+(\lambda_2-\lambda_4)\mu_6=0,\\ &(\lambda_5-\lambda_4)\mu_3+(\lambda_3-\lambda_5)\mu_4+(\lambda_4-\lambda_3)\mu_5=0. \end{aligned}$$

Put

$$\mathcal{A}=\begin{pmatrix} \lambda_3-\lambda_2 & \lambda_1-\lambda_3 & \lambda_2-\lambda_1 & 0 & 0 & 0\\ \lambda_6-\lambda_5 & 0 & 0 & 0 & \lambda_1-\lambda_6 & \lambda_5-\lambda_1\\ 0 & \lambda_4-\lambda_6 & 0 & \lambda_6-\lambda_2 & 0 & \lambda_2-\lambda_4\\ 0 & 0 & \lambda_5-\lambda_4 & \lambda_3-\lambda_5 & \lambda_4-\lambda_3 & 0 \end{pmatrix}.$$

The number of solutions of this system is equal to the number of columns of the matrix $\mathcal{A}$ minus the rank of $\mathcal{A}$. If $rk\mathcal{A} = 4$, we have two solutions : $\mu_i = 1$ lead to the invariant H_2 and $\mu_i = \lambda_i$ lead to the invariant H_3. This is unacceptable. If $rk\mathcal{A} = 3$, each four-order minor of $\mathcal{A}$ is singular. Now

$$\begin{pmatrix} \lambda_3-\lambda_2 & \lambda_1-\lambda_3 & \lambda_2-\lambda_1 & 0 \\ \lambda_6-\lambda_5 & 0 & 0 & 0 \\ 0 & \lambda_4-\lambda_6 & 0 & \lambda_6-\lambda_2 \\ 0 & 0 & \lambda_5-\lambda_4 & \lambda_3-\lambda_5 \end{pmatrix} = -(\lambda_6-\lambda_5)\,C,$$

$$\begin{pmatrix} \lambda_1-\lambda_3 & \lambda_2-\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1-\lambda_6 \\ \lambda_4-\lambda_6 & 0 & \lambda_6-\lambda_2 & 0 \\ 0 & \lambda_5-\lambda_4 & \lambda_3-\lambda_5 & \lambda_4-\lambda_3 \end{pmatrix} = (\lambda_1-\lambda_6)\,C,$$

$$\begin{pmatrix} \lambda_2-\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1-\lambda_6 & \lambda_5-\lambda_1 \\ 0 & \lambda_6-\lambda_2 & 0 & \lambda_2-\lambda_4 \\ \lambda_5-\lambda_4 & \lambda_3-\lambda_5 & \lambda_4-\lambda_3 & 0 \end{pmatrix} = -(\lambda_2-\lambda_1)\,C,$$

where

$$\begin{aligned} C \equiv\; & \lambda_1\lambda_6\lambda_4+\lambda_1\lambda_2\lambda_5-\lambda_1\lambda_2\lambda_4+\lambda_3\lambda_6\lambda_5-\lambda_3\lambda_6\lambda_4-\lambda_3\lambda_2\lambda_5 \\ & +\lambda_4\lambda_2\lambda_5+\lambda_4\lambda_1\lambda_3-\lambda_4\lambda_1\lambda_5+\lambda_6\lambda_2\lambda_3-\lambda_6\lambda_2\lambda_5-\lambda_1\lambda_6\lambda_3, \end{aligned}$$

and it follows that the condition for which these minors are zero is $C = 0$. Notice that this relation holds by cycling the indices : $1 \curvearrowright 4, 2 \curvearrowright 5, \curvearrowright 3 \curvearrowright 6$. Under Manakov (Manakov 1976) conditions,

$$\begin{aligned} \lambda_1 &= \frac{\beta_2-\beta_3}{\alpha_2-\alpha_3},\ \lambda_2 = \frac{\beta_1-\beta_3}{\alpha_1-\alpha_3},\ \lambda_3 = \frac{\beta_1-\beta_2}{\alpha_1-\alpha_2}, \\ \lambda_4 &= \frac{\beta_1-\beta_4}{\alpha_1-\alpha_4},\ \lambda_5 = \frac{\beta_2-\beta_4}{\alpha_2-\alpha_4},\ \lambda_6 = \frac{\beta_3-\beta_4}{\alpha_3-\alpha_4}, \end{aligned} \tag{21}$$

where $\alpha_i, \beta_i \in \mathbb{C}$, $\prod_{i<j}(\alpha_i-\beta_j) \neq 0$, equations (20) admits a Lax equation with an indeterminate h :

$$\begin{aligned} & \dot{\overbrace{(X+\alpha h)}} = [X+\alpha h, \Lambda X+\beta h], \\ \alpha &= \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \\ 0 & 0 & \beta_3 & 0 \\ 0 & 0 & 0 & \beta_4 \end{pmatrix}. \end{aligned} \tag{22}$$

$\Updownarrow$

$$\begin{aligned} \dot{X} &= [X, \Lambda.X] \Leftrightarrow (19), \\ [X,\beta] &+ [\alpha, \Lambda.X] = 0 \Leftrightarrow (21), \\ [\alpha,\beta] &= 0 \text{ trivially satisfied for diagonal matrices.} \end{aligned}$$

The parameters $\mu_1,\ldots,\mu_6$ can be parameterized (like $\lambda_1,\ldots,\lambda_6$) by :

$$\mu_1=\frac{\gamma_2-\gamma_3}{\alpha_2-\alpha_3},\ \mu_2=\frac{\gamma_1-\gamma_3}{\alpha_1-\alpha_3},\ \mu_3=\frac{\gamma_1-\gamma_2}{\alpha_1-\alpha_2}$$

$$\mu_4=\frac{\gamma_1-\gamma_4}{\alpha_1-\alpha_4},\ \mu_5=\frac{\gamma_2-\gamma_4}{\alpha_2-\alpha_4},\ \mu_6=\frac{\gamma_3-\gamma_4}{\alpha_3-\alpha_4}.$$

To use the method of isospectral deformations, consider the Kac-Moody extension $(n=4)$: $\mathcal{L}=\left\{\sum_{-\infty}^{N}A_ih^i : N\text{arbitrary } \in\mathbb{Z}, A_i\in gl(n,\mathbb{R})\right\}$, of $gl(n,\mathbb{R})$ with the bracket:

$$\left[\sum A_ih^i,\sum B_jh^j\right]=\sum_k\left(\sum_{i+j=k}[A_i,B_j]\right)h^k,$$

and the ad-invariant form: $\langle\sum A_ih^i,\sum B_jh^j\rangle=\sum_{i+j=-1}\langle A_i,B_j\rangle$, where $\langle,\rangle$ is the usual form defined on $gl(n,\mathbb{R})$. Let $\mathcal{K}$ and $\mathcal{N}$ be respectively the ≥ 0 and <0 powers of h in $\mathcal{L}$, then $\mathcal{L}=\mathcal{K}+\mathcal{N}$, for the pairing defined above $\mathcal{K}=\mathcal{K}^\perp$, $\mathcal{N}=\mathcal{N}^\perp$, so that $\mathcal{K}=\mathcal{N}^*$. The orbits described in this way come equipped with a symplectic structure with Poisson bracket $\{H_1,H_2\}(\alpha)=\langle\alpha,[\nabla_{\mathcal{K}^*}H_1,\nabla_{\mathcal{K}^*}H_2]\rangle$, where $\alpha\in\mathcal{K}^*$ and $\nabla_{\mathcal{K}^*}H\in\mathcal{K}$. According to the Adler-Kostant-Symes theorem (Lesfari 1999), the flow (22) is hamiltonian on an orbit through the point $X+ah$, $X\in so(4)$) formed by the coadjoint action of the subgroup $G_{\mathcal{N}}\subset SL(n)$ of lower triangular matrices on the dual Kac-Moody algebra $\mathcal{N}^*\approx\mathcal{K}^\perp=\mathcal{K}$. As a consequence, the coefficients of z^ih^i appearing in curve :

$$\Gamma:\ \left\{(z,h)\in\mathbb{C}^2:\det(X+ah-zI)=0\right\},\tag{23}$$

associated to the equation (22), are invariant of the system in involution for the symplectic structure of this orbit. Notice that

$$\det\left(gXg^{-1}\right)=\det X=(x_1x_4+x_2x_5+x_3x_6)^2,$$

$$tr\left(gXg^{-1}\right)^2=tr\left(gX^2g^{-1}\right)=tr\left(X^2\right)=-2\left(x_1^2+x_2^2+\cdots+x_6^2\right).$$

Also the complex flows generated by these invariants can be realized as straight lines on the abelian variety defined by the periods of curve Γ. Explicitly, equation (23) looks as follows

$$\Gamma:\quad\prod_{i=1}^{4}(\alpha_ih-z)+2H_4h^2-2H_1zh+2H_2z^2+H_3^2=0,\tag{24}$$

where $H_1(X)=c_1,\ H_2(X)=c_2,\ H_3(X)=2H=c_3,\ H_4(X)=c_4$. with c_1,c_2,c_3,c_4 generic constants. Γ is a curve of genus 3 and it has a natural involution $\sigma:\Gamma\to\Gamma$, $(z,h)\mapsto(-z,-h)$. Therefore the jacobian variety $Jac(\Gamma)$ of Γ splits up into an even and old part : the even part is an elliptic curve $\Gamma_0=\Gamma/\sigma$ and the odd part is a $2-$dimensional abelian surface $Prym(\Gamma/\Gamma_0)$ called the Prym variety : $Jac(\Gamma)=\Gamma_0+Prym_\sigma(\Gamma)$.. The van Moerbeke-Mumford linearization method (van Moerbeke and Mumford 1979) provides then an algebraic map from the complex affine variety $\bigcap_{i=1}^4\{H_i(X)=c_i\}\subset\mathbb{C}^6$ to the Jacobi variety $Jac(\Gamma)$. By the antisymmetry of Γ , this map sends this variety to the Prym variety $Prym_\sigma(\Gamma)$:

$$\bigcap_{i=1}^{4}\{H_i(X)=c_i\}\to Prym_\sigma(\Gamma)\ ,\ p\mapsto\sum_{k=1}^{3}s_k,$$

and the complex flows generated by the constants of the motion are straight lines on $Prym_\sigma(\Gamma)$. Finally, we have the

Proposition 2.2 The geodesic flow (19) is a hamiltonian system with

$$H \equiv H_1 = \frac{1}{2}\left(\lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_6 x_6^2\right),$$

the hamiltonian. It has two trivial invariants

$$\begin{aligned} H_2 &= \frac{1}{2}\left(x_1^2 + x_2^2 + \cdots + x_6^2\right), \\ H_3 &= x_1 x_4 + x_2 x_5 + x_3 x_6. \end{aligned}$$

Moreover, if

$$\lambda_1\lambda_6\lambda_4 + \lambda_1\lambda_2\lambda_5 - \lambda_1\lambda_2\lambda_4 + \lambda_3\lambda_6\lambda_5 - \lambda_3\lambda_6\lambda_4 - \lambda_3\lambda_2\lambda_5$$

$$+\lambda_4\lambda_2\lambda_5 + \lambda_4\lambda_1\lambda_3 - \lambda_4\lambda_1\lambda_5 + \lambda_6\lambda_2\lambda_3 - \lambda_6\lambda_2\lambda_5 - \lambda_1\lambda_6\lambda_3 = 0,$$

the system (19) has a fourth independent constant of the motion of the form

$$H_4 = \frac{1}{2}\left(\mu_1 x_1^2 + \mu_2 x_2^2 + \cdots + \mu_6 x_6^2\right).$$

Then the system (19) is completely integrable and can be linearized on the Prym variety $Prym_\alpha(\Gamma)$.

Example 2.1 The Kirchhoff's equations of motion of a solid in an ideal fluid have the form

$$\dot{p}_1 = p_2\frac{\partial H}{\partial l_3} - p_3\frac{\partial H}{\partial l_2}, \qquad \dot{l}_1 = p_2\frac{\partial H}{\partial p_3} - p_3\frac{\partial H}{\partial p_2} + l_2\frac{\partial H}{\partial l_3} - l_3\frac{\partial H}{\partial l_2},$$

$$\dot{p}_2 = p_3\frac{\partial H}{\partial l_1} - p_1\frac{\partial H}{\partial l_3}, \qquad \dot{l}_2 = p_3\frac{\partial H}{\partial p_1} - p_1\frac{\partial H}{\partial p_3} + l_3\frac{\partial H}{\partial l_1} - l_1\frac{\partial H}{\partial l_3},$$

$$\dot{p}_3 = p_1\frac{\partial H}{\partial l_2} - p_2\frac{\partial H}{\partial l_1}, \qquad \dot{l}_3 = p_1\frac{\partial H}{\partial p_2} - p_2\frac{\partial H}{\partial p_1} + l_1\frac{\partial H}{\partial l_2} - l_2\frac{\partial H}{\partial l_1},$$

where (p_1, p_2, p_3) is the velocity of a point fixed relatively to the solid, (l_1, l_2, l_3) the angular velocity of the body expressed with regard to a frame of reference also fixed relatively to the solid and H is the hamiltonian. These equations can be regarded as the equations of the geodesics of the right-invariant metric on the group $E(3) = SO(3) \times \mathbb{R}^3$ of motions of 3-dimensional euclidean space $\mathbb{R}^3$, generated by rotations and translations. Hence the motion has the trivial coadjoint orbit invariants $\langle p, p\rangle$ and $\langle p, l\rangle$. As it turns out, this is a special case of a more general system of equations written as

$$\dot{x} = x \wedge \frac{\partial H}{\partial x} + y \wedge \frac{\partial H}{\partial y}, \quad \dot{y} = y \wedge \frac{\partial H}{\partial x} + x \wedge \frac{\partial H}{\partial y},$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ et $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The first set can be obtained from the second by putting $(x, y) = (l, p/\varepsilon)$ and letting $\varepsilon \to 0$. The latter set of equations is the geodesic flow on $SO(4)$ for a left invariant metric defined by the quadratic form H.

2.3 The Toda Lattice

The Toda lattice equations (discretized version of the Korteweg-de Vries equation; in short K-dV equation : $\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$. This is an infinite-dimensional completely integrable system) motion of n particles with exponential restoring forces are governed by the following hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N} e^{q_i - q_{i+1}}, \qquad q_{N+1} = q_1.$$

The hamiltonian equations can be written as follows

$$\dot{q}_i = p_i, \quad \dot{p}_i = -e^{q_i - q_{i+1}} + e^{q_{i-1} - q_i}.$$

In term of the Flaschka's variables (Flaschka 1974a; Flaschka 1974b): $a_i = \frac{1}{2}e^{q_i - q_{i+1}}$, $b_i = -\frac{1}{2}p_i$, Toda's equations take the following form

$$\dot{a}_i = a_i\left(b_{i+1} - b_i\right), \quad \dot{b}_i = 2(a_i^2 - a_{i-1}^2), \tag{25}$$

with $b_{N+1} = b_1$ and $a_0 = a_N$. To show that the system (25) is completely integrable, one should find N first integrals independent and in involution each other. From the second equation, we have

$$\frac{d}{dt}\sum_{i=1}^{N} b_i = \sum_{i=1}^{N} \frac{db_i}{dt} = 0,$$

and we normalize the b_i's by requiring that $\sum_{i=1}^{N} b_i = 0$. This is a first integral for the system. We further define $N \times N$ matrices A and B with

$$A = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_N \\ a_1 & b_2 & \vdots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & b_{N-1} & a_{N-1} \\ a_N & \cdots & 0 & a_{N-1} & b_N \end{pmatrix}, B = \begin{pmatrix} 0 & a_1 & \cdots & \cdots & -a_N \\ -a_1 & 0 & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{N-1} \\ a_N & \cdots & \cdots & -a_{N-1} & 0 \end{pmatrix}.$$

Then (25) is equivalent to the Lax equation

$$\dot{A} = [B, A].$$

From proposition 1.5, we know that the quantities $I_k = \frac{1}{k}trA^k$, $1 \leq k \leq N$, are first integrals of motion : To be more precise

$$\dot{I}_k = tr(\dot{A}.A^{k-1}) = tr([B, A].A^{k-1}) = tr(BA^k - ABA^{k-1}) = 0.$$

Notice that I_1 is the first integral already know. Since these N first integrals are shown to be independent and in involution each other, the system (25) is thus completely integrable.

2.4 The Garnier Potential

Consider the hamiltonian

$$H=\frac{1}{2}\left(x_1^2+x_2^2\right)-\frac{1}{2}\left(\lambda_1 y_1^2+\lambda_2 y_2^2\right)+\frac{1}{4}\left(y_1^2+y_2^2\right)^2, \tag{26}$$

where λ_1 and λ_2 are constants. The corresponding system is given by

$$\begin{aligned} \dot{y}_1 &= x_1, \qquad \dot{x}_1=\left(\lambda_1-y_1^2-y_2^2\right)y_1, \\ \dot{y}_2 &= x_2, \qquad \dot{x}_2=\left(\lambda_2-y_2^2-y_1^2\right)y_2. \end{aligned} \tag{27}$$

Proposition 2.3 The system (27) has the additional first integral

$$\begin{aligned} H_2 &= \frac{1}{4}\left((x_1y_2-x_2y_1)^2-\left(\lambda_2 y_1^4+\lambda_1 y_2^4\right)-(\lambda_1+\lambda_2)\,y_1^2y_2^2\right) \\ &\quad +\frac{1}{2}\left(\lambda_1\lambda_2\left(y_1^2+y_2^2\right)-\left(\lambda_2 x_1^2+\lambda_1 x_2^2\right)\right). \end{aligned}$$

and is completely integrable. The flows generated by $H_1=H$(26) and H_2 are straight line motions on the jacobian variety of a smooth genus two hyperelliptic curve $\mathcal{H}$(29) associated to a Lax equation.

Proof. We consider the Lax representation in the form $\dot{A}=[A,B]$, with the following ansatz for the Lax operator

$$A=\begin{pmatrix} U & V \\ W & -U \end{pmatrix}, \qquad B=\begin{pmatrix} 0 & 1 \\ R & 0 \end{pmatrix}$$

where

$$\begin{aligned} V &= -(h-\lambda_1)(h-\lambda_2)(1+\frac{1}{2}(\frac{y_1^2}{h-\lambda_1}+\frac{y_2^2}{h-\lambda_2})), \\ U &= \frac{1}{2}(h-\lambda_1)(h-\lambda_2)(\frac{x_1y_1}{h-\lambda_1}+\frac{x_2y_2}{h-\lambda_2}), \\ W &= (h-\lambda_1)(h-\lambda_2)(\frac{1}{2}(\frac{x_1^2}{h-\lambda_1}+\frac{x_2^2}{h-\lambda_2})-h+\frac{1}{2}(y_1^2+y_2^2)), \\ R &= h-y_1^2-y_2^2. \end{aligned} \tag{28}$$

We form the curve in (z,h) space

$$P(h,z)=\det\left(A-zI\right)=0,$$

whose coefficients are functions of the phase space. Explicitly, this equation looks as follows

$$\begin{aligned} \mathcal{H}: z^2 &= P_5\left(h\right), \\ &= (h-\lambda_1)(h-\lambda_2)(h^3-(\lambda_1+\lambda_2)h^2+(\lambda_1\lambda_2-H_1)h-H_2), \end{aligned} \tag{29}$$

with H_1 (26) the hamiltonian and a second quartic integral H_2 of the form

$$\begin{aligned} H_2 &= -\frac{1}{4}(\lambda_2 y_1^4 + \lambda_1 y_2^4 + (\lambda_1 + \lambda_2) y_1^2 y_2^2 - (x_1 y_2 - x_2 y_1)^2) \\ &\quad -\frac{1}{2}(\lambda_2 x_1^2 + \lambda_1 x_2^2 - \lambda_1 \lambda_2 (y_1^2 + y_2^2)). \end{aligned}$$

The functions H_1 and H_2 commute :

$$\{H_1, H_2\} = \sum_{k=1}^{2} \left(\frac{\partial H_1}{\partial x_k} \frac{\partial H_2}{\partial y_k} - \frac{\partial H_1}{\partial y_k} \frac{\partial H_2}{\partial x_k} \right) = 0,$$

and the system (27) is completely integrable. The curve $\mathcal{H}$ determined by the fifth-order equation (29) is smooth, hyperelliptic and its genus is 2. Obviously, $\mathcal{H}$ is invariant under the hyperelliptic involution $(h, z) \curvearrowright (h, -z)$. Using the van Moerbeke-Mumford linearization method (van Moerbeke and Mumford 1979), we show that the linearized flow could be realized on the jacobian variety $Jac\,(\mathcal{H})$ of the genus 2 curve $\mathcal{H}$. For generic $c = (c_1, c_2) \in \mathbb{C}^2$ the affine variety defined by

$$M_c = \bigcap_{i=1}^{2} \left\{ x \in \mathbb{C}^4 : H_i\,(x) = c_i \right\},$$

is a smooth affine surface. According to the schema of (Eilbeck *et al.* 1993) and (Eilbeck *et al.* 1994), we introduce coordinates s_1 and s_2 on the surface M_c, such that $M_c\,(s_i) = 0, \lambda_1 \neq \lambda_2$, i.e.,

$$s_1 + s_2 = \frac{1}{2}\left(y_1^2 + y_2^2\right) + \lambda_1 + \lambda_2, \quad s_1 s_2 = \frac{1}{2}\left(\lambda_2 y_1^2 + \lambda_1 y_2^2\right) + \lambda_1 \lambda_2.$$

After some algebraic manipulations, we obtain the following equations for s_1 and s_2 :

$$\dot{s}_1 = 2\frac{\sqrt{P_5\,(s_1)}}{s_1 - s_2}, \quad \dot{s}_2 = 2\frac{\sqrt{P_5\,(s_2)}}{s_2 - s_1},$$

where $P_5\,(s)$ is defined by (29) . These equations can be integrated by the abelian mapping

$$\mathcal{H} \longrightarrow Jac\,(\mathcal{H}) = \mathbb{C}^2/L\ ,\ p \longmapsto \left(\int_{p_0}^{p} \omega_1\ ,\ \int_{p_0}^{p} \omega_2 \right),$$

where the hyperelliptic curve $\mathcal{H}$ of genus two is given by the equation (29), L is the lattice generated by the vectors $n_1 + \Omega n_2, (n_1, n_2) \in \mathbb{Z}^2, \Omega$ is the matrix of period of the curve $\mathcal{H}$, (ω_1, ω_2) is a canonical basis of holomorphic differentials on $\mathcal{H}$, i.e.,

$$\omega_1 = \frac{ds}{\sqrt{P_5\,(s)}}\ , \quad \omega_2 = \frac{sds}{\sqrt{P_5\,(s)}},$$

and p_0 is a fixed point.

2.5 The Coupled Nonlinear Schrödinger Equations

The system of two coupled nonlinear Schrödinger equations is given by

$$\begin{aligned} i\frac{\partial a}{\partial z}+\frac{\partial^2 a}{\partial t^2}+\Omega_0 a+\frac{2}{3}\left(|a|^2+|b|^2\right)a+\frac{1}{3}\left(a^2+b^2\right)\overline{a} &= 0, \\ i\frac{\partial b}{\partial z}+\frac{\partial^2 b}{\partial t^2}-\Omega_0 b+\frac{2}{3}\left(|a|^2+|b|^2\right)b+\frac{1}{3}\left(a^2+b^2\right)\overline{b} &= 0, \end{aligned} \tag{30}$$

where $a\,(z,t)$ and $b\,(z,t)$ are functions of z and t, the bar "$-$" denotes the complex conjugation, "$||$" denotes the modulus and Ω_0 is a constant. These equations play a significant role in mathematics, with a important number of physical applications. We seek solutions of (30) in the following form

$$a\,(z,t)=y_1\,(t)\exp\left(i\Omega z\right), \quad b\,(z,t)=y_2\,(t)\exp\left(i\Omega z\right),$$

where $y_1\,(t)$ et $y_2\,(t)$ are two functions and Ω is an arbitrary constant. Then we obtain the system

$$\ddot{y}_1+\left(y_1^2+y_2^2\right)y_1=\left(\Omega-\Omega_0\right)y_1,$$

$$\ddot{y}_2+\left(y_1^2+y_2^2\right)y_2=\left(\Omega+\Omega_0\right)y_2.$$

The latter coincides obviously with (27) for $\lambda_1=\Omega-\Omega_0$ and $\lambda_2=\Omega+\Omega_0$.

2.6 The Yang-Mills Equations

We consider the Yang-Mills system for a field with gauge group $SU(2)$:

$$\nabla_j F_{jk}=\frac{\partial F_{jk}}{\partial \tau_j}+[A_j,F_{jk}]=0,$$

where $F_{jk},A_j\in T_eSU(2), 1\leq j,k\leq 4$ and $F_{jk}=\frac{\partial A_k}{\partial \tau_j}-\frac{\partial A_j}{\partial \tau_k}+[A_j,A_k]$. The self-dual Yang-Mills (SDYM) equations is an universal system for which some reductions include all classical tops from Euler to Kowalewski (0+1-dimensions), K-dV, Nonlinear Schrdinger, Sine-Gordon, Toda lattice and N-waves equations (1+1-dimensions), KP and D-S equations (2+1-dimensions). In the case of homogeneous double-component field, we have $\partial_j A_k=0, j\neq 1, A_1=A_2=0, A_3=n_1U_1\in su(2), A_4=n_2U_2\in su(2)$ where n_i are $su(2)$-generators (i.e., they satisfy commutation relations : $n_1=[n_2,[n_1,n_2]], n_2=[n_1,[n_2,n_1]]$). The system becomes

$$\frac{\partial^2 U_1}{\partial t^2}+U_1U_2^2=0, \qquad \frac{\partial^2 U_2}{\partial t^2}+U_2U_1^2=0,$$

with $t=\tau_1$. By setting $U_j=q_j$, $\frac{\partial U_j}{\partial t}=p_j$, $j=1,2$, Yang-Mills equations are reduced to hamiltonian system

$$\dot{x}=J\frac{\partial H}{\partial x}, \quad x=(q_1,q_2,p_1,p_2)^{\intercal}, \quad J=\begin{pmatrix} O & -I \\ I & O \end{pmatrix},$$

with $H=\frac{1}{2}(p_1^2+p_2^2+q_1^2q_2^2)$, the hamiltonian. The symplectic transformation $p_1\curvearrowleft\frac{\sqrt{2}}{2}(p_1+p_2), p_2\curvearrowleft\frac{\sqrt{2}}{2}(p_1-p_2), q_1\curvearrowleft\frac{1}{2}(\sqrt[4]{2})(q_1+iq_2), q_2\curvearrowleft\frac{1}{2}(\sqrt[4]{2})(q_1-iq_2)$, takes this hamiltonian into

$$H=\frac{1}{2}(p_1^2+p_2^2)+\frac{1}{4}q_1^4+\frac{1}{4}q_2^4+\frac{1}{2}q_1^2q_2^2,$$

which coincides with (27) for $\lambda_1 = \lambda_2 = 0$.

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