A CONNECTION BETWEEN GEOMETRY AND DYNAMICAL SYSTEMS

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Abstract. This paper deals with a geometric and systematic approach to the integration of a nonlinear dynamical system: the anisotropic harmonic oscillator in a radial quartic potential. We study this system from a different angle: *a*) We show, using a Lax-type representation of the Hamilton's equations of motion, that the system is linearized in the jacobian variety of a smooth genus 2 hyperelliptic curve. *b*) We find via Kowalewski-Painlevé analysis the principal balances of the hamiltonian vector field defined by the hamiltonian and we show that the system is algebraic complete integrable. *c*) We also describe an explicit embedding of the abelian variety which completes the generic invariant surface, into projective space. *d*) We give a direct proof that the abelian variety obtained in this paper is dual to Prym variety and can also be seen as a double unramified cover of the jacobian variety of an hyperelliptic curve of genus 2. *e*) We show that at some special values of the parameters λ_1 and λ_2, we can describe elliptic solutions which are associated with two-gap elliptic solitons of the Korteweg-de Vries equation.

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1. Introduction

A symplectic variety (M, ω) is given by a $2n$-dimensional variety having a closed non-degenerate differential 2-form ω. Let H be a C^∞-function on M, called the hamiltonian. A hamiltonian vector field X_H or hamiltonian system is the dynamical system given by $\omega(X_H, v) = \langle dH, v\rangle$ for all vectors $v \in TM$ (tangent bundle of M). Such a hamiltonian system is called completely integrable if there exist n integrals $H_1 = H, H_2, \ldots, H_n$ in involution (i.e., such that the associated Poisson bracket $\{H_i, H_j\} = \omega(X_{H_i}, X_{Hj})$ all vanish) with linearly independent gradients

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(i.e., $dH_1 \wedge ... \wedge dH_n \neq 0$). For generic $c = (c_1, ..., c_n)$ the level set

$$\mathcal{A}_c = \{x : H_1(x) = c_1, \ldots , H_n(x) = c_n\},$$

is a smooth variety. When $\mathcal{A}_c$ is compact and connected, it is diffeomorphic to an n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ and each vector field will define a linear flow there. To be precise, in some open neighbourhood of the torus one can introduce regular symplectic coordinates $s_1, \ldots , s_n, \varphi_1, \ldots , \varphi_n$ in which ω takes the canonical form $\omega = ds_1 \wedge d\varphi_1 + \cdots + ds_n \wedge d\varphi_n$. Here the functions s_i (called action-variables) give coordinates in the direction transverse to the torus and can be expressed functionally in terms of the integrals H_i. The functions φ_i (called angle-variables) give standard angular coordinates on the torus, and every vector field X_{Hi} can be written in the form $\dot{\varphi_i} = h_i (s_1, \ldots , s_n)$, that is, its integral trajectories define a conditionally-periodic motion on the torus. Consequently, in a neighbourhood of the torus the hamiltonian vector field X_{Hi} take the following form $\dot{s_i} = 0$, $\dot{\varphi_i} = h_i (s_1, \ldots , s_n)$ and can be integrated by quadratures. All what precedes is the content of what is to-day usually called the Arnold-Liouville theorem [22] .
In this paper, we consider a 2-dimensional anisotropic harmonic oscillator in a radial quartic potential which is defined [29] by

$$H = \frac{1}{2}\left(\sum_{k=1}^{2}(x_k^2 - \lambda_k y_k^2)\right) + \frac{1}{4}\left(\sum_{k=1}^{2} y_k^2\right)^2 . \tag{1}$$

The corresponding system is given by

$$\begin{aligned} \dot{y}_1 &= x_1, & \dot{x}_1 &= \left(\lambda_1 - y_1^2 - y_2^2\right) y_1, \\ \dot{y}_2 &= x_2, & \dot{x}_2 &= \left(\lambda_2 - y_2^2 - y_1^2\right) y_2. \end{aligned} \tag{2}$$

The integrability of the system (2) has been studied by several authors [6] , [28] , [29] and others. In section 2, we give a Lax representation of the flow generated by (1) which depends on a spectral parameter and we prove that the system (2) is completely integrable. In section 3, we make a careful study of the algebraic geometric aspects of the level varieties of this integrable system. We find via Kowalewski-Painlevé analysis the principal balances of the hamiltonian vector field defined by the hamiltonian and we show that the system is algebraic complete integrable. We also describe an explicit embedding of the abelian variety which completes the generic invariant surface, into projective space. In section 4, we give a direct proof that the abelian variety obtained in section 3 is dual to Prym variety and can also be seen as a double unramified cover of the jacobian variety of an hyperelliptic curve of genus 2. The purpose of section 5, is to show that at some special values of the parameters λ_1 and λ_2, we can describe elliptic solutions which are associated with two-gap elliptic solitons of the Korteweg-de Vries equation.
Among the results presented in this paper, there is an explicit calculation of invariants for a hamiltonian system which cut out an open set in an abelian variety, and various curves related to this abelian variety are given explicitly. The integrable dynamical system presented here is an interesting example, particular to experts of abelian varieties who may want to see an explicit example of a correspondence for varieties defined by different curves.

2. Lax representation and complete integrability

New examples of completely integrable hamiltonian systems, which have recently been discovered, are based on the Lax representation of the equations of motion. By a Lax equation with a parameter h, we shall mean an equation

$$\dot{A}(h) = [B(h), A(h)] \equiv B(h)A(h) - A(h)B(h), \tag{3}$$

where

$$A(h) = \sum_{-p}^{q} A_k(t)h^k, \qquad B(h) = \sum_{-p}^{q} B_k(t)h^k,$$

are finite Laurent series in a variable h whose coefficients are matrices depending on a parameter. Some hamiltonian flows on Kostant-Kirillov coadjoint orbits in subalgebras of infinite dimensional Lie algebras (Kac-Moody Lie algebras) yield large classes of extended Lax pairs (3). A general statement leading to such situations is given by the Adler-Kostant-Symes theorem [17]. Using the van Moerbeke-Mumford linearization method [24], Adler and van Moerbeke [1] showed that the linearized flow could be realized on the jacobian variety $Jac(\Gamma)$ (or some sub-abelian variety of it) of an algebraic curve Γ (spectral curve) associated to (3). Recall that the jacobian variety of Γ is a complex torus $Jac\,(\Gamma) = \mathbb{C}^g / \{n_1 + \Omega n_2\}$ where g is the genus of Γ, $(n_1, n_2 \in \mathbb{Z}^g)$, $\Omega = \left(\int_{\gamma_i} \omega_j\right)$, $1 \le i, j \le g$, $(\omega_1, \dots, \omega_g)$ a base of holomorphic differentials on Γ and $\gamma_i \in H_1(\Gamma, \mathbb{Z})$.

A hamiltonian flow of the type (3) preserves the spectrum of A and therefore its characteristic polynomial $P(z,h) = \det(A(h) - zI)$. The curve $\Gamma : P(z,h) = 0$, of genus g, is time independent, i.e., its coefficients are integrals of the motion (3). We then construct an algebraic map from the complex invariant varieties of these hamiltonian systems to the jacobian variety $Jac(\Gamma)$ of the curve Γ. Therefore all the complex flows generated by the constants of the motion are straight line motions on this Jacobian variety, i.e., the linearizing equations are given by

$$\sum_{i=1}^{g} \int_{s_i(0)}^{s_i(t)} \omega_k = c_k t \ , \ 0 \le k \le g,$$

where $\omega_1, \dots, \omega_g$ span the g-dimensional space of holomorphic differentials on the curve Γ of genus g.

According to the schema of [7, 8, 9, 25], we consider the Lax representation of the equations (2) in the form

$$\dot{A}(h) = [B(h), A(h)], \tag{4}$$

with the following ansatz for the Lax operator

$$A(h) = \begin{pmatrix} v(h) & u(h) \\ w(h) & -v(h) \end{pmatrix}, \qquad B(h) = \begin{pmatrix} 0 & 1 \\ r(h) & 0 \end{pmatrix},$$

where

$$u(h) = -1 - \frac{1}{2}\left(\frac{y_1^2}{h-\lambda_1} + \frac{y_2^2}{h-\lambda_2}\right), \qquad v(h) = -\frac{1}{2}\dot{u}(h),$$

$$w(h) = \frac{1}{2}\left(\frac{x_1^2}{h-\lambda_1} + \frac{x_2^2}{h-\lambda_2}\right) - h + \frac{1}{2}(y_1^2 + y_2^2), \qquad r(h) = h - y_1^2 - y_2^2.$$

Consider the algebraic curve

$$\Gamma : \det(A(h) - zI) = 0, \tag{5}$$

whose coefficients are functions of the phase space. Using the van Moerbeke-Mumford linearization method [24, 18, 20], we show that the linearized flow could be realized on the jacobian variety $Jac(\Gamma)$ of the curve Γ (5) associated to (4). A hamiltonian flow of the type (4) preserves the spectrum of $A(h)$ and therefore its characteristic polynomial $\det(A(h) - zI)$. The curve Γ(5)is time independent, i.e., its coefficients are integrals of the motion (4). Explicitly, equation (5) looks as follows

$$\Gamma : w^2 = P_5(h), \tag{6}$$

where $w = (h - \lambda_1)(h - \lambda_2)z$, $P_5(\lambda)$ is a polynomial of degree 5 of the form

$$P_5(h) = (h^3 - (\lambda_1 + \lambda_2)h^2 + (\lambda_1\lambda_2 - H_1)h - H_2)(h - \lambda_1)(h - \lambda_2),$$

$H_1 = H$ is defined by (1) with λ_1, λ_2 arbitrary and a second quartic integral H_2 of the form

$$\begin{aligned} H_2 &= \frac{1}{4}((x_1y_2 - x_2y_1)^2 - (\lambda_2y_1^4 + \lambda_1y_2^4) - (\lambda_1 + \lambda_2)y_1^2y_2^2) \\ &\quad + \frac{1}{2}(\lambda_1\lambda_2(y_1^2 + y_2^2) - (\lambda_2x_1^2 + \lambda_1x_2^2)). \end{aligned} \tag{7}$$

The curve Γ determined by the fifth-order equation (6) is smooth, hyperelliptic and its genus is 2. Obviously, Γ is invariant under the hyperelliptic involution $(h, w) \mapsto (h, -w)$. The second hamiltonian vector field is cubic and is written as

$$\begin{aligned} \dot{y}_1 &= \frac{1}{2}(x_1y_2 - x_2y_1)y_2 - \lambda_2x_1, \\ \dot{y}_2 &= -\frac{1}{2}(x_1y_2 - x_2y_1)y_1 - \lambda_1x_2, \\ \dot{x}_1 &= \frac{1}{2}(x_1y_2 - x_2y_1)x_2 + \lambda_2y_1^3 + \frac{1}{2}(\lambda_1 + \lambda_2)y_1y_2^2 - \lambda_1\lambda_2y_1, \\ \dot{x}_2 &= -\frac{1}{2}x_1(x_1y_2 - x_2y_1)x_1 + \lambda_1y_2^3 + \frac{1}{2}(\lambda_1 + \lambda_2)y_1^2y_2 - \lambda_1\lambda_2y_2. \end{aligned} \tag{8}$$

These vector fields are in involution with respect to the associated Poisson bracket. For generic $c = (c_1, c_2) \in \mathbb{C}^2$ the affine variety defined by

$$\mathcal{A}_c = \bigcap_{i=1}^{2} \{x : H_i(x) = c_i\} \subset \mathbb{C}^4, \tag{9}$$

is the fibre of a morphism from $\mathbb{C}^4$ to $\mathbb{C}^2$ and then $\mathcal{A}_c$ is a smooth affine surface. For $\lambda_1 = \lambda_2$, it is easy to show that the problem can be integrated in terms of elliptic functions. For $\lambda_1 \neq \lambda_2$, we introduce coordinates s_1 and s_2 on the surface $\mathcal{A}_c$ by setting

$$\begin{aligned} s_1 + s_2 &= \frac{1}{2}(y_1^2 + y_2^2) + \lambda_1 + \lambda_2, \\ s_1s_2 &= \frac{1}{2}(\lambda_2y_1^2 + \lambda_1y_2^2) + \lambda_1\lambda_2. \end{aligned} \tag{10}$$

After some algebraic manipulations, we obtain the following equations for s_1 and s_2 :

$$\begin{aligned} \dot{s}_1 = \frac{ds_1}{dt} = 2\frac{\sqrt{P_5(s_1)}}{s_1 - s_2}, \\ \dot{s}_2 = \frac{ds_2}{dt} = 2\frac{\sqrt{P_5(s_2)}}{s_2 - s_1}, \end{aligned}$$

where $P_5(s)$ is defined by (6). These equations can be integrated by the abelian mapping

$$\Gamma \longrightarrow Jac(\Gamma) = \mathbb{C}^2/\Lambda, \quad (p_1, p_2) \mapsto (\xi_1, \xi_2),$$

where the hyperelliptic curve Γ of genus 2 is given by the equation (6), Λ is the lattice generated by the vectors $n_1 + \Omega n_2, (n_1, n_2) \in \mathbb{Z}^2, \Omega$ is the matrix of period of the curve Γ, (ω_1, ω_2) is a canonical basis of holomorphic differentials on Γ, i.e.,

$$\omega_1 = \frac{ds}{\sqrt{P_5(s)}}, \qquad \omega_2 = \frac{sds}{\sqrt{P_5(s)}},$$

$p_1 = (s_1, \sqrt{P_5(s_1)})$, $p_2 = (s_2, \sqrt{P_5(s_2)})$, p_0 is a fixed point and

$$\xi_1 = \int_{p_0}^{p_1} \omega_1 + \int_{p_0}^{p_2} \omega_1,$$

$$\xi_2 = \int_{p_0}^{p_1} \omega_2 + \int_{p_0}^{p_2} \omega_2.$$

We have

$$\dot{\xi}_1 = \frac{d\xi_1}{dt} = \frac{\dot{s}_1}{\sqrt{P_5(s_1)}} + \frac{\dot{s}_2}{\sqrt{P_5(s_2)}} = 0,$$

$$\dot{\xi}_2 = \frac{d\xi_2}{dt} = \frac{s_1\dot{s}_1}{\sqrt{P_5(s_1)}} + \frac{s_2\dot{s}_2}{\sqrt{P_5(s_2)}} = 2,$$

and hence the problem can be integrated in terms of genus 2 hyperelliptic functions of time. Consequently, we have

Theorem 1. *The hamiltonian system* (2) *is completely integrable for all* λ_1, λ_2 *and admits a Lax representation given by* (4). *The invariants in this case are quartic: the first integral is given by the hamiltonian* $H_1 = H(1)$ *whereas the second* H_2 *has the form* (7). *For* $\lambda_1 = \lambda_2$, *the problem can be integrated in terms of elliptic functions. For* $\lambda_1 \neq \lambda_2$, *the flows generated by* H_1 *and* H_2 *are straight line motions on the jacobian variety* $Jac(\Gamma)$ *of a smooth genus* 2 *hyperelliptic curve* Γ (6) *associated to Lax equation* (4).

3. Linearizing flow on complex algebraic tori

Consider hamiltonian problems of the form

$$X_H : \dot{x} = J\frac{\partial H}{\partial x}\ , \ x \in \mathbb{R}^m, \tag{11}$$

where H is the hamiltonian and $J = J(x)$ is a skew-symmetric matrix with polynomial entries in x, for which the corresponding Poisson bracket

$$\{H_i, H_j\} = \langle \partial H_i/\partial x, J\partial H_j/\partial x\rangle\,,$$

satisfies the Jacobi identities. The system (11) with polynomial right hand side will be called algebraic complete integrable (a.c.i.) when [2, 11, 13] :
a) The system possesses $n + k$ independent polynomial invariants $H_1, ..., H_{n+k}$ (Casimir functions) of which k lead to zero vector fields $J\frac{\partial H_{n+i}}{\partial x}(x) = 0$, $1 \leq i \leq k$, the n remaining ones are in involution (i.e., $\{H_i, H_j\} = 0$) and $m = 2n + k$. For most values of $c_i \in \mathbb{R}$, the invariant varieties $\bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m : H_i = c_i\}$ are assumed

compact and connected. Then, according to the Arnold-Liouville theorem, there exists a diffeomorphism

$$\bigcap_{i=1}^{n+k}\{x\in\mathbb{R}^m : H_i=c_i\}\to\mathbb{R}^n/Lattice,$$

and the solutions of the system (11) are straight lines motions on these tori.
b) The invariant varieties, thought of as affine varieties in $\mathbb{C}^m$ can be completed into complex algebraic tori, i.e.,

$$\bigcap_{i=1}^{n+k}\{H_i=c_i, x\in\mathbb{C}^m\}\cup\mathcal{D}=\mathbb{C}^n/Lattice,$$

where $\mathbb{C}^n/Lattice$ is a complex algebraic torus (i.e., abelian variety)and $\mathcal{D}$ a divisor (i.e. one or several codimension one subvarieties). Algebraic means that the torus can be defined as an intersection $\bigcap_{i=1}^{M}\{P_i(X_0,...,X_N)=0\}$ involving a large number of homogeneous polynomials P_i. In the natural coordinates $(t_1,...,t_n)$ of $\mathbb{C}^n/Lattice$ coming from $\mathbb{C}^n$, the functions $x_i=x_i(t_1,...,t_n)$ are meromorphic and (11) defines straight line motion on $\mathbb{C}^n/Lattice$. Condition b) means, in particular, there is an algebraic map $(x_1(t),...,x_m(t))\mapsto(s_1(t),...,s_n(t))$ making the following sums linear in t :

$$\sum_{i=1}^{n}\int_{s_i(0)}^{s_i(t)}\omega_j=\mu_j t\ ,\ 1\le j\le n,\ \mu_j\in\mathbb{C},$$

where $\omega_1,...,\omega_n$ denote holomorphic differentials on some algebraic curves.
Adler and van Moerbeke [2] have shown that the existence of a coherent set of Laurent solutions :

$$x_i=\sum_{j=0}^{\infty}x_i^{(j)}t^{j-k_i},\quad k_i\in\mathbb{Z},\quad \text{some } k_i>0,$$

depending on $dim\ (phase\ space)-1=m-1$ free parameters is necessary and sufficient for a hamiltonian system with the right number of constants of motion to be a.c.i. So, if the hamiltonian flow (11) is a.c.i., it means that the variables x_i are meromorphic on the torus $\mathbb{C}^n/Lattice$ and by compactness they must blow up along a codimension one subvariety (a divisor) $\mathcal{D}\subset\mathbb{C}^n/Lattice$. By the a.c.i. definition, the flow (11) is a straight line motion in $\mathbb{C}^n/Lattice$ and thus it must hit the divisor $\mathcal{D}$ in at least one place. Moreover through every point of $\mathcal{D}$, there is a straight line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equations must admit Laurent expansions which depend on the $n-1$ parameters defining $\mathcal{D}$ and the $n+k$ constants c_i defining the torus $\mathbb{C}^n/Lattice$, the total count is therefore $m-1=dim\ (phase\ space)-1$ parameters. For others aspects concerning a.c.i. systems, see [12, 14, 15, 16, 19, 21, 23].
The system (11) can be written in the form (11) with $m=4$, $n=2$ and $k=0$. To be more precise

$$X_{H_1}:\quad \dot{x}=J\frac{\partial H}{\partial x}\ ,\quad x=(y_1,y_2,x_1,x_2)^\top, \tag{12}$$

where

$$\frac{\partial H}{\partial x} = (\frac{\partial H}{\partial y_1}, \frac{\partial H}{\partial y_2}, \frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2})^\top, \qquad H = H_1(1), \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The second hamiltonian vector field commuting with the first is regulated by the equations

$$X_{H_2}: \quad \dot{x} = J\frac{\partial H_2}{\partial x}, \qquad x = (y_1, y_2, x_1, x_2)^\top, \tag{13}$$

with H_2 defined by (7) and is explicitly given by (8).
In section 2, we have shown that the problem can be integrated in terms of genus 2 hyperelliptic functions. In fact , the transformation to the separating coordinates s_1 and s_2 which leads to the quadratures in terms of hyperelliptic integrals is quite involved. Finding this transformation require a great deal of luck and ingenuity. So, the question is how does one prove directly (i.e., without any reference to a trick of isospectral deformation with an indeterminate h) that the system is effectively a.c.i. with abelian space coordinates? The idea of the direct proof we shall give here is closely related to the geometric spirit of the (real) Arnold-Liouville theorem discussed in section 1. Namely, a compact complex n-dimensional variety on which there exist n holomorphic commuting vector fields which are independent at every point is analytically isomorphic to a n-dimensional complex torus $\mathbb{C}^n/Lattice$ and the complex flows generated by the vector fields are straight lines on this complex torus. Now, the affine surface $\mathcal{A}_c(9)$ is not compact and the main problem will be to complete $\mathcal{A}_c$ into a non singular compact complex algebraic variety $\widetilde{\mathcal{A}}_c = \mathcal{A}_c \cup \mathcal{D}$ in such a way that the vector fields X_{H_1} and X_{H_2} extend holomorphically along the divisor $\mathcal{D}$ and remain independent there. If this is possible, $\widetilde{\mathcal{A}}_c$ is an algebraic complex torus, i.e., an abelian variety and the coordinates x_i restricted to $\mathcal{A}_c$ are abelian functions. A naive guess would be to take the natural compactification $\overline{\mathcal{A}}_c$ of $\mathcal{A}_c$ by projectivizing the equations :

$$\overline{\mathcal{A}}_c = \bigcap_{i=1}^{2} \{X : H_i(X) = c_i X_0^4\} \subset \mathbb{P}^4(\mathbb{C}).$$

Indeed, this can never work for a general reason: an abelian variety $\widetilde{\mathcal{A}}_c$ of dimension bigger or equal than two is never a complete intersection, that is it can never be described in some projective space $\mathbb{P}^n(\mathbb{C})$ by n-dim $\widetilde{\mathcal{A}}_c$ global polynomial homogeneous equations. In other words, if $\mathcal{A}_c$ (9) is to be the affine part of an abelian surface, $\overline{\mathcal{A}}_c$ must have a singularity somewhere along the locus at infinity $\overline{\mathcal{A}}_c \cap \{X_0 = 0\}$. In fact, we shall show that the existence of meromorphic solutions to the differential equations (2) depending on 3 free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor. Here are the main steps of the method :
STEP 1 : Consider Laurent series solutions

$$y_i = \sum_{j=0}^{\infty} y_i^{(j)} t^{j-1}, \qquad x_i = \dot{y}_i, \qquad i = 1, 2 \tag{14}$$

which depend on 3 free parameters α, β, γ. Putting (14) into the differential equations (2), one finds at the 0^{th} step a non-linear system

$$\begin{aligned} 2y_1^{(0)} &= -(y_1^{(0)})^3 - y_1^{(0)}(y_2^{(0)})^2, \\ 2y_2^{(0)} &= -(y_2^{(0)})^3 - (y_1^{(0)})^2 y_2^{(0)}, \end{aligned} \tag{15}$$

and at the k^{th} step, a system of linear equations

$$(L - kI)x^{(k)} = \begin{cases} 0 & \text{for} \quad k = 1, \\ \text{a polynomial in } x^{(1)}, ..., x^{(k)} & \text{for} \quad k \geq 1, \end{cases} \tag{16}$$

where L is the Jacobian matrix of the equations (15). The parameter α appear at the 0^{th} step, i.e., in the resolution of (15) and the 2 remaining ones β, γ at the k^{th} step, $k = 3, 4$, in the resolution of (16). The Laurent series solutions (14) are explicitly given by

$$\begin{aligned} (17) y_1 &= \frac{\alpha}{t} + \frac{1}{6}((\lambda_2 - \lambda_1)\alpha^2 + 2\lambda_2 - 3\lambda_1)\alpha t + \beta t^2 \\ &\quad -\alpha(\frac{1}{24}(\lambda_2 - \lambda_1)(2(\lambda_2 - \lambda_1)\alpha^2 - 3\lambda_1 + \lambda_2) + \frac{\varepsilon i \gamma}{\sqrt{(2 + \alpha^2)}})t^3 + \cdots , \\ y_2 &= \frac{\varepsilon i \sqrt{2 + \alpha^2}}{t} + \frac{\varepsilon i}{6}((\lambda_2 - \lambda_1)\alpha^2 - \lambda_2)\sqrt{2 + \alpha^2} t + \frac{\varepsilon i \alpha \beta}{\sqrt{2 + \alpha^2}} t^2 + \gamma t^3 + \cdots , \end{aligned}$$

and $x_1 = \dot{y}_1$, $x_2 = \dot{y}_2$ with $\varepsilon \equiv \pm 1$.

STEP 2: By substituting these series in the constants of the motion $H_1 = c_1$ and $H_2 = c_2$, one eliminates the parameter γ linearly, leading to algebraic relation between the two remaining parameters, which is nothing but the equation of the divisor $\mathcal{D}$ along which the y_1, y_2, x_1, x_2 blow up. The Laurent solutions are parameterized by two copies C_ε ($\varepsilon = \pm 1$)of the same genus 3 hyperelliptic curve C defined by

$$C : \quad \beta^2 = (2 + \alpha^2)(A_1\alpha^6 + A_2\alpha^4 + A_3\alpha^2 + A_4), \tag{18}$$

where $\lambda \equiv \lambda_1 - \lambda_2$, $A_1 \equiv \lambda^3/72$, $A_2 \equiv \lambda^2 (2\lambda_1 - \lambda_2)/36$, $A_3 \equiv \lambda (\lambda\lambda_1 - c_1)/18$, $A_4 \equiv -(\lambda_1 c_1 + c_2)/9$. Moreover, the map

$$\sigma : C \longrightarrow C, \; (\alpha, \beta) \mapsto (-\alpha, \beta), \tag{19}$$

is an involution on C and the quotient $C_0 = C/\sigma$ is an elliptic curve defined by

$$C_0 : \beta^2 = (2 + \zeta)\left(A_1\zeta^3 + A_1\zeta^2 + A_3\zeta + A_4\right). \tag{20}$$

The curve C is a double ramified covering of C_0,

$$\varphi : C \longrightarrow C_0, (\alpha, \beta) \mapsto (\zeta, \beta), \tag{21}$$

ramified at the four points covering $\zeta = 0$ and ∞. The curve C can also be seen as a 2−sheeted unramified cover

$$\pi : C \longrightarrow \Gamma, (\alpha, \beta) \mapsto (\zeta, \eta), \tag{22}$$

of the following hyperelliptic curve Γ of genus 2 :

$$\Gamma : \eta^2 = \zeta (2 + \zeta)\left(A_1\zeta^3 + A_1\zeta^2 + A_3\zeta + A_4\right). \tag{23}$$

STEP 3: As motivation, let us first recall some basics concepts. Let $\widetilde{\mathcal{A}}_c$ be a smooth variety. A divisor $\mathcal{D}$ on $\widetilde{\mathcal{A}}_c$ is a formal sum $\mathcal{D} = \sum n_i V_i$, $n_i \in \mathbb{Z}$, of irreducible analytic hypersurfaces on $\widetilde{\mathcal{A}}_c$. The set of all divisors on $\widetilde{\mathcal{A}}_c$ forms an abelian group.

Two divisors may be added or subtracted performing the corresponding operation on the coefficients n_i of the corresponding V_i. A divisor $\mathcal{D}$ is said to be effective or positive if each $n_i \geq 0$. Two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ are linearly equivalent if their difference $\mathcal{D}_1 - \mathcal{D}_2$ is the divisor of a meromorphic function f, denoted by (f), i.e., $(f) = (f)_0 - (f)_\infty =$ (zero divisor) $-$ (pole divisor). Here we denote also by $\widetilde{\mathcal{A}}_c$ a smooth variety compactifying $\mathcal{A}_c(9)$ and $\mathcal{D}$ the embedding of $C_{+1} + C_{-1}$ into $\mathbb{P}^N(\mathbb{C})$. Consider a basis $1, f_1, ..., f_N$ of the vector space

$$L(\mathcal{D}) \equiv \{f : \ f \text{ meromorphic on } \widetilde{\mathcal{A}}_c \text{ such that } (f) \geq -\mathcal{D} \},$$

of meromorphic functions on $\widetilde{\mathcal{A}}_c$ with at worst a simple pole along $\mathcal{D}$ and the map

$$\widetilde{\mathcal{A}}_c \longrightarrow \mathbb{P}^N(\mathbb{C}), \ p \mapsto [1, f_1(p), ..., f_N(p)],$$

considered projectively, because if at p some $f_i(p) = \infty$, we divide by f_i having the highest order pole near p, which makes every element finite. The Kodaira embedding theorem tells us that if the line bundle associated with the divisor is positive, then for $k \in \mathbb{N}$, the functions of $L(k\mathcal{D})$ (i.e., meromorphic functions on $\widetilde{\mathcal{A}}_c$ having a k-fold pole at worst along $\mathcal{D}$) embed smoothly $\widetilde{\mathcal{A}}_c$ into $\mathbb{P}^N(\mathbb{C})$ and then by Chow's theorem,$\widetilde{\mathcal{A}}_c$ can be realized as an algebraic variety. In fact in our case, $k = 2$ suffices, i.e., the divisor $2\mathcal{D}$ provides a smooth embedding into $\mathbb{P}^{15}(\mathbb{C})$, via the meromorphic sections of $L(2\mathcal{C})$. Put $\mathcal{S} \equiv 2\mathcal{D}$ and let $\chi(\mathcal{S}) = \dim L(\mathcal{S})$ be the Euler characteristic of $\mathcal{S}$. The adjunction formula and the Riemann-Roch theorem for divisors on abelian surfaces (see [10]) imply that

$$g(\mathcal{S}) = \frac{1}{2}\left(K_{\widetilde{\mathcal{A}}_c}.\mathcal{S} + \mathcal{S}.\mathcal{S}\right) + 1,$$

$$\chi(\mathcal{S}) = p_a\left(\widetilde{\mathcal{A}}_c\right) + 1 + \frac{1}{2}\left(\mathcal{S}.(\mathcal{S}\backslash K_{\widetilde{\mathcal{A}}_c})\right),$$

where $g(\mathcal{S})$ is the geometric genus of $\mathcal{S}$, $p_a\left(\widetilde{\mathcal{A}}_c\right)$ is the arithmetic genus of $\widetilde{\mathcal{A}}_c$, $K_{\widetilde{\mathcal{A}}_c}$ is the canonical divisor on $\widetilde{\mathcal{A}}_c$ (i.e., the zero-locus of a holomorphic 2-form) and $\mathcal{S}.\mathcal{S}$ denote the number of intersection points of $\mathcal{S}$ with $a + \mathcal{S}$ where $a + \mathcal{S}$ is a small translation by a of $\mathcal{S}$ on $\widetilde{\mathcal{A}}_c$. For an abelian surface, we have $K_{\widetilde{\mathcal{A}}_c} = 0$, $p_a(\widetilde{\mathcal{A}}_c) = -1$ and

$$g(\mathcal{S}) - 1 = \frac{\mathcal{S}.\mathcal{S}}{2} = \chi(\mathcal{S}).$$

Using Kodaira-Serre duality $[10, p.153]$, Kodaira-Nakano vanishing theorem $[10, p.154]$ and a theorem on theta-functions $[10, p.317]$, it easy to see that

$$g(\mathcal{S}) - 1 = \dim L(\mathcal{S}) = N + 1 = \delta_1\delta_2, \tag{24}$$

where $\delta_1, \delta_2 \in \mathbb{N}^*$, $\delta_1 \mid \delta_2$, are the elementary divisors of the polarization of $\widetilde{\mathcal{A}}_c$. Recall that a Kähler variety is a variety with a Kähler metric, i.e., a hermitian metric whose associated differential 2-form of type $(1,1)$ is closed. The complex torus $\mathbb{C}^2/lattice$ with the euclidean metric $\sum dz_i \otimes d\overline{z}_i$ is a Kähler variety and any compact complex variety that can be embedded in projective space is also a Kähler variety. Now, a compact complex Kähler variety having as many independent meromorphic functions as its dimension is a projective variety.

Based on this motivation, it is easy to find a set of polynomial functions $\{1, f_1, ..., f_N\}$ in $L(\mathcal{S})$, i.e., $L(2\mathcal{D})$ such that the embedding of $\mathcal{S}$ with those functions into $\mathbb{P}^N(\mathbb{C})$

yields a curve of genus $N+2$. Straightforward calculation, using asymptotic expansions, shows that the space $L(S)$ is spanned by the following functions

$$L(\mathcal{S}) = \{1, f_1, ..., f_{15}\}, \tag{25}$$

where

$$f_1 = y_1 = \frac{\alpha}{t} + o(t), \qquad f_2 = y_2 = \frac{\varepsilon i\sqrt{2+\alpha^2}}{t} + o(t),$$

$$f_3 = x_1 = -\frac{\alpha}{t^2} + o(t), \qquad f_4 = x_2 = -\frac{\varepsilon i\sqrt{2+\alpha^2}}{t^2} + o(t),$$

$$f_5 = f_1^2 = \frac{\alpha^2}{t^2} + o(t), \qquad f_6 = f_2^2 = -\frac{2+\alpha^2}{t^2} + o(t),$$

$$f_7 = f_1 f_2 = i\alpha\varepsilon\frac{\sqrt{(2+\alpha^2)}}{t^2} + o(t),$$

$$f_8 = f_1 f_4 - f_2 f_3 = i\alpha\varepsilon\sqrt{(2+\alpha^2)}\frac{\lambda_1 - \lambda_2}{t} + o(t),$$

$$f_9 = f_1 f_8 = -i\frac{\alpha^2}{t^2}\varepsilon\sqrt{(2+\alpha^2)}(\lambda_2 - \lambda_1) + o(t),$$

$$f_{10} = f_2 f_8 = \frac{2+\alpha^2}{t^2}\alpha(\lambda_2 - \lambda_1) + o(t),$$

$$f_{11} = f_8^2 = -\alpha^2\left(2+\alpha^2\right)\frac{(\lambda_2 - \lambda_1)^2}{t^2} + o(t),$$

$$f_{12} = (f_5 + f_6) f_8 - 2(\lambda_1 f_1 f_4 - \lambda_2 f_2 f_3) = \frac{12 i\varepsilon\beta}{t^2\sqrt{(2+\alpha^2)}} + o(t),$$

$$f_{13} = (f_5 + f_6) f_1 f_2 + 2 f_3 f_4 = -i\alpha\varepsilon\sqrt{(2+\alpha^2)}\frac{(\lambda_2 - \lambda_1)\alpha^2 + 2\lambda_1}{t^2} + o(t),$$

$$f_{14} = f_3 f_8 - (\lambda_2 - \lambda_1) f_1 f_7 = \frac{6 i\alpha\varepsilon\beta}{t^2\sqrt{(2+\alpha^2)}} + o(t),$$

$$f_{15} = f_4 f_8 - (\lambda_2 - \lambda_1) f_2 f_7 = -\frac{6\beta}{t^2} + o(t).$$

Using these functions $1, f_1, ..., f_{15}$, one embeds each of the curves C_{+1} and C_{-1} into $\mathbb{P}^{15}(\mathbb{C})$. Thus embedded, they have two points in commune $s_1 \equiv \left(\alpha = i\sqrt{2}, \beta = 0\right)$ and $s_2 \equiv \left(\alpha = -i\sqrt{2}, \beta = 0\right)$, at which they are tangent to each other. The divisor $\mathcal{D} = C_{+1} + C_{-1}$ obtained in this way has genus 5 and thus $\mathcal{S} = 2\mathcal{D}$ has genus 17, satisfying the requirement (24). Now we need to attaches the affine part of the intersection (9) of the two invariants so as to obtain a smooth compact connected surface in $\mathbb{P}^{15}(\mathbb{C})$. To be precise, the orbits of the vector field (12) running through $\mathcal{S}$ form a smooth surface Σ near $\mathcal{S}$ such that $\Sigma \setminus \mathcal{S} \subseteq \widetilde{\mathcal{A}}_c$ and the variety $\widetilde{\mathcal{A}}_c = \mathcal{A}_c \cup \Sigma$ is smooth, compact and connected. Indeed, let

$$\psi(t,p) = \{x(t) = (y_1(t), y_2(t), x_1(t), x_2(t)) : t \in \mathbb{C}, 0\langle |t| \langle \varepsilon\},$$

be the orbit of the vector field (12) going through the point $p \in \mathcal{S}$. Let $\Sigma_p \subset \mathbb{P}^{15}(\mathbb{C})$ be the surface element formed by the divisor $\mathcal{S}$ and the orbits going through p, and set $\Sigma \equiv \bigcup_{p\in\mathcal{S}} \Sigma_p$. Consider the curve $\mathcal{S}' = \mathcal{H} \cap \Sigma$ where $\mathcal{H} \subset \mathbb{P}^{15}(\mathbb{C})$ is a hyperplane transversal to the direction of the flow. If $\mathcal{S}'$ is smooth, then using the implicit function theorem the surface Σ is smooth. But if $\mathcal{S}'$ is singular at 0, then Σ would be singular along the trajectory ($t-$axis) which go immediately into the affine part $\widetilde{\mathcal{A}}_c$. Hence, $\widetilde{\mathcal{A}}_c$ would be singular which is a contradiction because $\mathcal{A}_c$ is the fibre of a morphism from $\mathbb{C}^4$ to $\mathbb{C}^2$ and so smooth for almost all the two constants of the motion c_i. Next, let $\overline{\mathcal{A}}_c$ be the projective closure of $\mathcal{A}_c$ into $\mathbb{P}^4(\mathbb{C})$, let $X = (X_0, X_1, X_2, Y_1, Y_2) \in \mathbb{P}^4(\mathbb{C})$ and let $I = \overline{\mathcal{A}}_c \cap \{X_0 = 0\}$ be the locus at infinity. Consider the map

$$\overline{\mathcal{A}}_c \subseteq \mathbb{P}^4(\mathbb{C}) \longrightarrow \mathbb{P}^{15}(\mathbb{C}),\ X \mapsto f(X),$$

where $f = (1, f_1, ..., f_{15}) \in L(\mathcal{S})$ and let $\widetilde{\mathcal{A}}_c = f(\overline{\mathcal{A}}_c)$. In a neighbourhood $V(p) \subseteq \mathbb{P}^{15}(\mathbb{C})$ of p, we have $\Sigma_p = \widetilde{\mathcal{A}}_c$ and $\Sigma_p \setminus \mathcal{S} \subseteq \mathcal{A}_c$. Otherwise there would exist an element of surface $\Sigma_p' \subseteq \widetilde{\mathcal{A}}_c$ such that

$$\Sigma_p \cap \Sigma_p' = t - axis,$$
$$orbit \quad \psi(t,p) = t - axis \setminus p \subseteq \mathcal{A}_c,$$

and hence $\mathcal{A}_c$ would be singular along the $t-$axis which is impossible. Since the variety $\overline{\mathcal{A}_c} \cap \{X_0 \neq 0\}$ is irreducible and since the generic hyperplane section $\mathcal{H}_{gen.}$ of $\overline{\mathcal{A}}_c$ is also irreducible, all hyperplane sections are connected and hence I is also connected. Now, consider the graph $\Gamma_f \subseteq \mathbb{P}^4(\mathbb{C}) \times \mathbb{P}^{15}(C)$ of the map f, which is irreducible together with $\overline{\mathcal{A}}_c$. It follows from the irreducibility of I that a generic hyperplane section $\Gamma_f \cap \{\mathcal{H}_{gen.} \times \mathbb{P}^{15}(\mathbb{C})\}$ is irreducible, hence the special hyperplane section $\Gamma_f \cap \{\{X_0 = 0\} \times \mathbb{P}^{15}(\mathbb{C})\}$ is connected and therefore the projection map

$$proj_{\mathbb{P}^{15}(\mathbb{C})}\{\Gamma_f \cap \{\{X_0 = 0\} \times \mathbb{P}^{15}(\mathbb{C})\}\} = f(I) \equiv \mathcal{S},$$

is connected. Hence, the variety $\mathcal{A}_c \cup \Sigma = \overline{\mathcal{A}}_c$ is compact, connected and embeds smoothly into $\mathbb{P}^{15}(\mathbb{C})$ via f.

STEP 4: We wish to show that $\widetilde{\mathcal{A}}_c$ is an abelian surface equipped with two everywhere independent commuting vector fields. For doing that, let g^{τ_1} and g^{τ_2} be the flows generated respectively by vector fields (12) and (13) . For $p \in \mathcal{S}$ and for small $\varepsilon > 0, \quad g^{\tau_1}(p), \forall \tau_1, 0 < |\tau_1| < \varepsilon,$ is well defined and $g^{\tau_1}(p) \in \mathcal{A}_c$. Then we may define g^{τ_2} on $\mathcal{A}_c$ by

$$g^{\tau_2}(q) = g^{-\tau_1} g^{\tau_2} g^{\tau_1}(q), \qquad q \in U(p) = g^{-\tau_1}(U(g^{\tau_1}(p))),$$

where $U(p)$ is a neighbourhood of p. By commutativity one can see that g^{τ_2} is independent of τ_1;

$$g^{-\tau_1 - \varepsilon_1} g^{\tau_2} g^{\tau_1 + \varepsilon_1}(q) = g^{-\tau_1} g^{-\varepsilon_1} g^{\tau_2} g^{\tau_1} g^{\varepsilon_1} = g^{-\tau_1} g^{\tau_2} g^{\tau_1}(q).$$

We affirm that $g^{\tau_2}(q)$ is holomorphic away from S. This because $g^{\tau_2} g^{\tau_1}(q)$ is holomorphic away from $\mathcal{S}$ and that g^{τ_1} is holomorphic in $U(p)$ and maps bi-holomorphically $U(p)$ onto $U(g^{\tau_1}(p))$. Now, since the flows g^{τ_1} and g^{τ_2} are holomorphic and independent on $\mathcal{S}$, we can show along the same lines as in the Arnold-Liouville theorem that $\widetilde{\mathcal{A}}_c$ is a complex torus $\mathbb{C}^2/lattice$ and so in particular $\widetilde{\mathcal{A}}_c$ is a Kähler variety. And that will done, by considering the local diffeomorphism

$$\mathbb{C}^2 \longrightarrow \widetilde{\mathcal{A}}_c,\ (\tau_1, \tau_2) \mapsto g^{\tau_1} g^{\tau_2}(p),$$

for a fixed origin $p \in \mathcal{A}_c$. The additive subgroup

$$\{(\tau_1, \tau_2) \in \mathbb{C}^2 : g^{\tau_1} g^{\tau_2}(p) = p\},$$

is a lattice of $\mathbb{C}^2$, hence $\mathbb{C}^2/lattice \to \widetilde{\mathcal{A}}_c$ is a biholomorphic diffeomorphism and $\widetilde{\mathcal{A}}_c$ is a Kähler variety with Kähler metric given by $d\tau_1 \otimes d\overline{\tau}_1 + d\tau_2 \otimes d\overline{\tau}_2$. As mentioned above, a compact complex Kähler variety having the required number as (its dimension) of independent meromorphic functions is a projective variety. In fact, here we have $\widetilde{\mathcal{A}}_c \subseteq \mathbb{P}^{15}(\mathbb{C})$ .Thus $\widetilde{\mathcal{A}}_c$ is both a projective variety and a complex torus $\mathbb{C}^2/lattice$ and hence an abelian surface as a consequence of Chow theorem. Finally, we have the

Theorem 2. *Let $\mathcal{A}_c$ (9) be the affine invariant variety defined by putting the two invariants* (1) *and* (7) *(with $\lambda_1 \neq \lambda_2$) of the flow* (2) *equal to generic constants, then*
a) The variety $\widetilde{\mathcal{A}}_c$ which completes $\mathcal{A}_c$ is a Kähler variety with Kähler metric given by $d\tau_1 \otimes d\overline{\tau}_1 + d\tau_2 \otimes d\overline{\tau}_2$.
b) $\mathcal{A}_c$ is the affine part of an abelian surface $\widetilde{\mathcal{A}}_c$ with $\widetilde{\mathcal{A}}_c \setminus \mathcal{A}_c = \mathcal{D}$ where $\mathcal{D} = \mathcal{C}_1 + \mathcal{C}_{-1}$ has genus 5 and consists of two copies $\mathcal{C}_\varepsilon$ $(\varepsilon = \pm 1)$ of the same genus 3 hyperelliptic curve $\mathcal{C}$ (18). The curves $\mathcal{C}_\varepsilon$ have two points in common, at which they are tangent to each other. The curve $\mathcal{C}$ is a double cover of an elliptic curve $\mathcal{C}_0$ (20) ramified at four points and can also be seen as a 2-sheeted unramified cover of an hyperelliptic curve Γ (23) of genus 2. Moreover, the hamiltonian flows generated by the vector fields X_{H_1} (12) and X_{H_2} (13) are straight lines on $\widetilde{\mathcal{A}}_c$.
c) The 16 functions: 1 , $f_1, ..., f_{15}$ (25) form a basis of the vector space $L(2\mathcal{D})$ of meromorphic functions on $\widetilde{\mathcal{A}}_c$ with at worst a double pole along $\mathcal{D}$. Moreover, the map

$$\widetilde{\mathcal{A}}_c \simeq \mathbb{C}^2/Lattice \longrightarrow \mathbb{P}^{15}(C) \ , \ (t_1, t_2) \mapsto [(1, f_1(t_1, t_2), ..., f_{15}(t_1, t_2))],$$

is an embedding of $\widetilde{\mathcal{A}}_c$ into $\mathbb{P}^{15}(\mathbb{C})$.

4. Abelian surface $\widetilde{\mathcal{A}}_c$ as Prym variety and jacobian variety

In the first part of this section, we shall show that the abelian surface $\widetilde{\mathcal{A}}_c$ can be identified as the dual of the Prym variety $Prym(\mathcal{C}/\mathcal{C}_0)$ on which the problem linearizes. In the second part, we show that $\widetilde{\mathcal{A}}_c$ can also be seen as a double unramified cover of the jacobian variety $Jac(\Gamma)$ of the 2-genus hyperelliptic curve Γ (23) and the problem is linearized in this jacobian variety. From the fundamental exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \overset{\exp}{\to} \mathcal{O}^* \to 0,$$

we get the map

$$\cdots \to H^1\left(\widetilde{\mathcal{A}}_c, \mathcal{O}\right) \to H^1\left(\widetilde{\mathcal{A}}_c, \mathcal{O}^*\right) \overset{c_1}{\to} H^2\left(\widetilde{\mathcal{A}}_c, \mathbb{Z}\right) \to H^2\left(\widetilde{\mathcal{A}}_c, \mathcal{O}\right) \to \cdots,$$

where $c_1(L)$ is the first Chern class of a line bundle L on $\widetilde{\mathcal{A}}_c$. Therefore the group $Pic^o\left(\widetilde{\mathcal{A}}_c\right)$ of holomorphic line bundles on $\widetilde{\mathcal{A}}_c$ with Chern class zero (any line bundle with Chern class zero can be realized by constant multipliers) is given by

$$Pic^o\left(\widetilde{\mathcal{A}}_c\right) = H^1\left(\widetilde{\mathcal{A}}_c, \mathcal{O}\right) \ / \ H^1\left(\widetilde{\mathcal{A}}_c, \mathbb{Z}\right) \simeq H^1\left(\widetilde{\mathcal{A}}_c, \mathcal{O}^*\right),$$

and is naturally isomorphic to the dual abelian surface $\widetilde{\mathcal{A}}_c^\vee$ of $\widetilde{\mathcal{A}}_c$ ($^\vee$ means the dual abelian surface). The relationship between $\widetilde{\mathcal{A}}_c$ and $\widetilde{\mathcal{A}}_c^\vee$ is symmetric like the relationship between two vectors spaces set up a bilinear pairing. Recall that a Prym variety $Prym(\mathcal{C}/\mathcal{C}_0)$ is a sub-abelian variety of the jacobian variety

$$Jac(\mathcal{C}) = Pic^0(\mathcal{C}) = H^1(\mathcal{O}_C)/H^1(\mathcal{C}, \mathbb{Z}),$$

constructed from the double cover φ (21) : the involution σ (19) on C (18) interchanging sheets, extends by linearity to a map $\sigma : Jac(\mathcal{C}) \longrightarrow Jac(\mathcal{C})$ and up to some points of order two, $Jac(\mathcal{C})$ splits into an even part and an odd part : the even part is an elliptic curve; the quotient of C by σ, i.e., C_0 (20) and the odd part is a 2−dimensional abelian surface $Prym(\mathcal{C}/\mathcal{C}_0)$.

Let $(a_0, b_0, a_1, b_1, a_2, b_2)$ be a canonical homology basis of C such that $\sigma(a_0) = a_1$, $\sigma(b_0) = b_1$, $\sigma(a_2) = -a_2$, $\sigma(b_2) = -b_2$ for the involution σ (19). As a basis of holomorphic differentials $\omega_0, \omega_1, \omega_2$ on the curve C (18) we take the differentials

$$\omega_0 = \frac{\alpha d\alpha}{\beta}, \qquad \omega_1 = \frac{\alpha^2 d\alpha}{\beta}, \qquad \omega_2 = \frac{d\alpha}{\beta}, \tag{26}$$

and obviously $\sigma^*(\omega_0) = \omega_0$, $\sigma^*(\omega_k) = -\omega_k$, $k = 1, 2$. We consider the period matrix Λ of $Jac\,(\mathcal{C})$

$$\Lambda = \begin{pmatrix} \omega_0(a_2) & \omega_0(b_2) & \omega_0(a_0) & \omega_0(b_0) & \omega_0(a_1) & \omega_0(b_1) \\ \omega_1(a_2) & \omega_1(b_2) & \omega_1(a_1) & \omega_1(b_1) & \omega_1(a_1) & \omega_1(b_1) \\ \omega_2(a_2) & \omega_2(b_2) & \omega_2(a_2) & \omega_2(b_2) & \omega_2(a_1) & \omega_2(b_1) \end{pmatrix},$$

where $\omega_k(a_j) = \int_{a_j} \omega_k$, $\omega_k(b_j) = \int_{b_j} \omega_k$, $0 \le k, j \le 2$, are the periods of ω_k over the cycles a_j, b_j, $0 \le j \le 2$. Note that $\omega_0(a_2) = \omega_0(b_2) = 0$ and $\omega_0(a_1) = \omega_0(\sigma(a_0)) = \sigma^*(\omega_0(a_0)) = \omega_0(a_0)$, $\omega_k(a_1) = \omega_k(\sigma(a_0)) = \sigma^*(\omega_k(a_0)) = -\omega_k(a_0)$, $k = 1, 2$, $\omega_0(b_1) = \omega_0(b_0)$, $\omega_k(b_1) = -\omega_k(b_0)$, $k = 1, 2$ and therefore

$$\Lambda = \begin{pmatrix} 0 & 0 & \omega_0(a_0) & \omega_0(b_0) & \omega_0(a_0) & \omega_0(b_0) \\ \omega_1(a_2) & \omega_1(b_2) & \omega_1(a_1) & \omega_1(b_1) & -\omega_1(a_0) & -\omega_1(b_0) \\ \omega_2(a_2) & \omega_2(b_2) & \omega_2(a_2) & \omega_2(b_2) & -\omega_2(a_0) & -\omega_2(b_0) \end{pmatrix}.$$

By elementary column operations, we obtain the following matrices

$$\Lambda = \begin{pmatrix} 0 & 0 & \omega_0(a_0) & \omega_0(b_0) & 2\omega_0(a_0) & 2\omega_0(b_0) \\ \omega_1(a_2) & \omega_1(b_2) & \omega_1(a_1) & \omega_1(b_1) & 0 & 0 \\ \omega_2(a_2) & \omega_2(b_2) & \omega_2(a_2) & \omega_2(b_2) & 0 & 0 \end{pmatrix},$$

and

$$\Lambda = \begin{pmatrix} 0 & 0 & \omega_0(a_0) & \omega_0(b_0) & 0 & 0 \\ \omega_1(a_2) & \omega_1(b_2) & \omega_1(a_1) & \omega_1(b_1) & 2\omega_1(a_0) & 2\omega_1(b_0) \\ \omega_2(a_2) & \omega_2(b_2) & \omega_2(a_2) & \omega_2(b_2) & 2\omega_2(a_0) & 2\omega_2(b_0) \end{pmatrix}.$$

The elliptic curve $\mathcal{C}_0$ and the Prym variety $Prym(\mathcal{C}/\mathcal{C}_0)$ have respective period matrices $(2\omega_0(a_0) \quad 2\omega_0(b_0))$ and

$$\Omega = \begin{pmatrix} \omega_1(a_2) & \omega_1(b_2) & 2\omega_1(a_0) & 2\omega_1(b_0) \\ \omega_2(a_2) & \omega_2(b_2) & 2\omega_2(a_0) & 2\omega_2(b_0) \end{pmatrix}. \tag{27}$$

Next we show, using the Laurent solutions, that the differentials dt_1 and dt_2 corresponding to the flows generated respectively by $H = H_1$ and H_2 , restricted to the curve $\mathcal{C}$ (18), descend to two differentials on $\mathcal{C}$:

$$\begin{aligned} dt_1|_{\mathcal{C}} &= \frac{\alpha^2 d\alpha}{\beta} = \omega_1, \\ dt_2|_{\mathcal{C}} &= \frac{d\alpha}{\beta} = \omega_2. \end{aligned} \tag{28}$$

Let $L_{\Omega^*} = \{\sum_{k=1}^{4} n_k \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix} (\nu_k) : n_k \in \mathbb{Z}\}$ be the lattice associated to the period matrix

$$\Omega^\vee = \begin{pmatrix} dt_1(\nu_1) & dt_1(\nu_2) & dt_1(\nu_3) & dt_1(\nu_4) \\ dt_2(\nu_1) & dt_2(\nu_2) & dt_2(\nu_3) & dt_2(\nu_4) \end{pmatrix},$$

where $(\nu_1, \nu_2, \nu_3, \nu_4)$ is a basis of $H_1(\widetilde{\mathcal{A}}_c, \mathbb{Z})$ and let

$$\widetilde{\mathcal{A}}_c \to \mathbb{C}^2 / L_{\Omega^*} : p \mapsto \int_{p_0}^{p} \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix},$$

be the uniformizing map. By the Lefschetz theorem on hyperplane section $[10, p.156]$, the map $H_1(\mathcal{C}, \mathbb{Z}) \longrightarrow H_1(\widetilde{\mathcal{A}}_c, \mathbb{Z})$ induced by the inclusion $\mathcal{C} \hookrightarrow \widetilde{\mathcal{A}}_c$ is surjective and consequently we can find 4 cycles $\nu_1, \nu_2, \nu_3, \nu_4$ on the curve $\mathcal{C}$ such that

$$\Omega^\vee = \begin{pmatrix} \omega_1(\nu_1) & \omega_1(\nu_2) & \omega_1(\nu_3) & \omega_1(\nu_4) \\ \omega_2(\nu_1) & \omega_2(\nu_2) & \omega_2(\nu_3) & \omega_2(\nu_4) \end{pmatrix},$$

and $L_{\Omega^*} = \left\{ \sum_{k=1}^{4} n_k \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} (\nu_k) : n_k \in \mathbb{Z} \right\}$. The cycles $\nu_1, \nu_2, \nu_3, \nu_4$ in C which we look for are a_2, b_2, a_0, b_0 and they generate $H_1(\widetilde{V}_c, \mathbb{Z})$ such that

$$\Omega^\vee = \begin{pmatrix} \omega_1(a_0) & \omega_1(b_0) & \omega_1(a_2) & \omega_1(b_2) \\ \omega_2(a_0) & \omega_2(b_0) & \omega_2(a_2) & \omega_2(b_2) \end{pmatrix}, \tag{29}$$

is a Riemann matrix. Otherwise there would exist a non trivial linear combination of the $\nu_1, ..., \nu_4$ with integer coefficients such that $ma_2 + nb_2 + la_0 + kb_0 = \partial \upsilon$, υ a $2-$chain on $\widetilde{\mathcal{A}}_c$. But dt_1 and dt_2 are closed differentials on $\widetilde{\mathcal{A}}_c$, so the Stokes' formula implies that

$$m \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} (a_2) + n \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} (b_2) + l \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} (a_0) + k \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} (b_0) = \begin{pmatrix} dt_1 \\ dt_2 \end{pmatrix} (\partial \upsilon) = 0,$$

which contradicts the linear independence of the columns of $\Omega^\vee$. Since Ω (27) is the period matrix of $Prym(\mathcal{C}/\mathcal{C}_0)$, then $\Omega^\vee$ (29) is the period matrix of the dual Prym variety $Prym(\mathcal{C}/\mathcal{C}_0)^\vee$. Consequently $\widetilde{\mathcal{A}}_c$ and $Prym(\mathcal{C}/\mathcal{C}_0)^\vee$ are two abelian varieties analytically isomorphic to the same complex torus $\mathbb{C}^2/L_{\Omega^*}$. By Chow's theorem, $\widetilde{\mathcal{A}}_c$ and $Prym(\mathcal{C}/\mathcal{C}_0)^\vee$ are then algebraically isomorphic. Using the differentials (26), the solutions of the problem reads

$$\int_{s_1(0)}^{s_1(t)} \omega_1 + \int_{s_2(0)}^{s_2(t)} \omega_1 = \mu_1 t,$$

$$\int_{s_1(0)}^{s_1(t)} \omega_2 + \int_{s_2(0)}^{s_2(t)} \omega_2 = \mu_2 t,$$

for some appropriate coordinates μ_1 and μ_2. Consequently, we have

Theorem 3. *The abelian surface $\widetilde{\mathcal{A}}_c$ which completes the affine surface $\mathcal{A}_c$ can be identified as:*

$$\widetilde{\mathcal{A}}_c = Prym(\mathcal{C}/\mathcal{C}_0)^\vee,$$

and the system (2) *or* (12) *linearizes on this variety, that is to say their solutions can be expressed in terms of abelian integrals.*

In section 2, we have shown that he system (12) can be integrated in terms of genus 2 hyperelliptic functions of time. Here we shall show (without reference to Lax representation as in section 2), using the results obtained in section 3 and starting from Vanhaecke's approach [27] , that the linearized flow can be realized on the jacobian variety $Jac\,(\Gamma)$ of the 2-genus hyperelliptic curve $\Gamma\,(23)$. Remember that the Laurent solutions (17) restricted to the surface $\mathcal{A}_c$ (9) are parameterized by two copies $\mathcal{C}_{+1}$ and $\mathcal{C}_{-1}$ of the same hyperelliptic curve $\mathcal{C}$ (18) of genus 3. The latter is a double ramified cover of an elliptic curve $\mathcal{C}_0$ (20) and a 2-1 unramified cover of an hyperelliptic curve Γ (23) of genus 2. The torus $\widetilde{\mathcal{A}}_c$ can also be regarded as a double unramified cover of the jacobian variety $Jac(\Gamma)$ and the system (12) can be integrated in terms of genus 2 hyperelliptic functions of time. The differentials dt_1 and dt_2, corresponding to the flows (12) and (13) , restricted to the curve C, go down to Γ. Indeed, using (28)

$$\begin{aligned} dt_1|_{\mathcal{C}} &= \omega_1 = \frac{\alpha^2 d\alpha}{\beta} = \frac{\zeta d\zeta}{\eta}, \\ dt_2|_{\mathcal{C}} &= \omega_2 = \frac{d\alpha}{\beta} = \frac{d\zeta}{\eta}, \end{aligned} \tag{30}$$

yielding the two hyperelliptic differentials on Γ. The map $\varphi\,(21)$ extends to a map $\widetilde{\varphi} : \widetilde{\mathcal{A}}_c \longrightarrow Jac(\Gamma)$. We denote by $|\mathcal{D}|$ the linear system, i.e., the set of all effective divisors linearly equivalent to $\mathcal{D}$. We have $|\mathcal{D}| = \mathbb{P}\,(L\,(\mathcal{D}))$; associating to each non-zero function $f \in L\,(\mathcal{D})$. Recall that an Abelian variety $\widetilde{\mathcal{A}}_c \simeq \mathbb{C}^m/Lattice$ has a natural involution τ, induced by the sign flip $(z_1, \dots , z_m) \mapsto (-z_1, \dots , -z_m)$ in $\mathbb{C}^m$. Its fixed points are exactly the 2^{2m} half-periods of $\widetilde{\mathcal{A}}_c$. The quotient $\widetilde{\mathcal{A}}_c/\tau$ is called the Kummer surface. In particular, the Kummer surface K of $Jac\,(\Gamma)$ can be considered as the image of $\psi_{\widetilde{\varphi}^*|2\Gamma|} : \widetilde{\mathcal{A}}_c \to \mathbb{P}^3\,(\mathbb{C})$ with $\widetilde{\varphi}^*\,|2\Gamma| \subset |2C|$. Since Γ is hyperelliptic of genus 2, it has 6 distinct Weierstrass points (Indeed, suppose $\mathcal{C}$ is a smooth hyperelliptic curve of genus $g \geq 2$ with $\varrho : \mathcal{C} \to \mathbb{P}^1\,(\mathbb{C})$ the two-sheeted map, then all the branch points of ϱ are the only Weierstrass points of $\mathcal{C}$. By the Riemann-Hurwitz formula, the number of these points is equal to $2g + 2$). Choose a Weierstrass point P on the curve Γ and coordinates $[Z_0, Z_1, Z_2, Z_3]$ for $\mathbb{P}^3\,(\mathbb{C})$ such that $\psi_{\widetilde{\varphi}^*|2\Gamma|}\,(P) = [0, 0, 0, 1]$. Then this point will be a singular point for the Kummer surface K of equation

$$a\,(Z_0, Z_1, Z_2)\;\; Z_3^2 + b\,(Z_0, Z_1, Z_2)\;\; Z_3 + c\,(Z_0, Z_1, Z_2)\;\; = 0,$$

where a, b and c are polynomials of degree respectively $2, 3$ and 4. After a projective transformation which fixes $[0, 0, 0, 1]$ we may assume that $a\,(Z_0, Z_1, Z_2) = Z_1^2 - 4Z_0Z_2$. We can construct an algebraic map from $\mathcal{A}_c$ to the Jacobi variety $Jac\,(\Gamma)$:

$$\mathcal{A}_c \to Jac\,(\Gamma)\,,\;\; p \in \mathcal{A}_c \mapsto (\zeta_1 + \zeta_2) \in Jac\,(\Gamma)\,,$$

and the flows generated by the constants of the motion are straight lines on $Jac\,(\Gamma)$, i.e., the linearizing equations are given by

$$\sum_{i=1}^{2} \int_{\zeta_i(0)}^{\zeta_i(t)} \omega_k = c_k t,\;\; 0 \leq k \leq 2,$$

where ω_1, ω_2 (30) span the 2-dimensional space of holomorphic differentials on the curve Γ and ζ_1, ζ_2 two appropriate variables given by

$$\zeta_1 = \frac{1}{2}\left(-Z_1 + \sqrt{a\left(Z_0, Z_1, Z_2\right)}\right),$$
$$\zeta_2 = \frac{1}{2}\left(-Z_1 - \sqrt{a\left(Z_0, Z_1, Z_2\right)}\right),$$

algebraically related to the originally given ones, for which the Hamilton-Jacobi equation could be solved by separation of variables.

5. Reduction to elliptic solutions

In this section, we show that at some special values of the parameters λ_1 and λ_2, the solutions for the integrable case proposed in this paper may be expressed in terms of elliptic functions. We solve the equations of motion with the help of finite-gap integration theory. The reader is referred to a recent and excellent book [5] for this theory and its applications to nonlinear integrable equations. For doing that, we need some notions about elliptic Weierstrass $\wp$-function which we summarize here after.

Let $\Lambda \subset \mathbb{C}$ be a lattice, which may be taken to be $\Lambda = \{2m\omega_1 + 2n\omega_2,\ (m,n) \in \mathbb{Z}^2\}$. We recall that the elliptic Weierstrass $\wp$-function with respect to Λ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{(m,n)\neq(0,0)\\(m,n)\in\mathbb{Z}^2}} \left\{\frac{1}{(z - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2}\right\},\ z \in \mathbb{C}.$$

This series converges absolutely and uniformly on every compact set in $\mathbb{C} \setminus \Lambda$, thus giving a meromorphic function having a double pole at each lattice point. This function is doubly periodic $\wp(z + 2m\omega_1 + 2n\omega_2) = \wp(z)$, $(m, n \in \mathbb{Z})$ with periods $2\omega_1, 2\omega_2$ such that $Im\frac{\omega_2}{\omega_1} > 0$ and has a singularity at $z = 0$ such as

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \frac{g_2^2}{1200}z^6 + \cdots,$$
$$g_2 = 60 \sum_{\substack{(m,n)\neq(0,0)\\(m,n)\in\mathbb{Z}^2}} \frac{1}{(2m\omega_1 + 2n\omega_2)^2},$$
$$g_3 = 140 \sum_{\substack{(m,n)\neq(0,0)\\(m,n)\in\mathbb{Z}^2}} \frac{1}{(2m\omega_1 + 2n\omega_2)^6}.$$

The derivative $\wp'(z)$ is again an elliptic function. The $\wp$-function satisfies the addition theorem

$$det\begin{pmatrix} 1 & \wp(a) & \wp'(a) \\ 1 & \wp(b) & \wp'(b) \\ 1 & \wp(c) & \wp'(c) \end{pmatrix} = 0, \qquad a + b + c = 0,$$

as well as the differential equation

$$\left(\wp'(z)\right)^2 = 4\wp^3(z) - g_2\wp(z) - g_3.$$

The map $\mathbb{C}/\Lambda \to \mathbb{P}^2(\mathbb{C})$, $z \mapsto [1, \wp(z), \wp'(z)]$, $z \neq 0$ and $0 \mapsto [0,0,1]$ is an isomorphism between the complex torus $\mathbb{C}/\Lambda$ and the elliptic curve

$$y^2 = 4x^3 - g_2x - g_3,$$

where $x = \wp(z)$ and $y = \wp'(z)$.
The elliptic solutions for which we look for are given by the spectral problem

$$\mathcal{L}\Psi = \lambda\Psi, \tag{31}$$

depending in the elliptic potential

$$\mathcal{U}(x) = 2\sum_{i=1}^{N} \wp(x - x_i) + C, \tag{32}$$

as

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} - \mathcal{U}(x).$$

Here $\Psi = \Psi(x, \lambda)$ is an eigenfunction (elliptic Backer-Akhiezer function) of operator $\mathcal{L}$, $\wp$ is the elliptic Weierstrass function defined above, C is a constant and $x_1, ..., x_N$ are on the locus

$$\Delta = \left\{ (x_1, ..., x_N) \in \mathbb{C}^N : \sum_{i \neq j} \wp'(x_i - x_j) = 0,\ x_i \neq x_j,\ j = 1, ..., N \right\}.$$

The geometry of the locus Δ was studied by several authors (see [3] for example). It was shown that Δ is non-empty if N is a triangular number, i.e., $N = g(g+1)/2$, where g is the number of gaps (or lacunas) in the spectrum or the genus of the associated hyperelliptic curve. It is shown in [3] that if $x_i = x_i(t)$, $j = 1, ..., N$ evolve according to the law

$$\dot{x}_i = -12\sum_{i \neq j} \wp(x_i - x_j),$$

then the function (32) is an elliptic solution of the well-known Korteweg-de Vries equation:

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

and is connected with the completely integrable Calogero-Moser system described by the hamiltonian:

$$H = \frac{1}{2}\sum_{i=1}^{N} y_i^2 - 2\sum_{i \neq j} \wp(x_i - x_j),$$

with y_i, x_i, $i = 1, ..., N$ being canonical variables.
Let us consider the two-gap potential for equation (31) normalized by its expansion near $x = 0$ as

$$\mathcal{U}(x) = \frac{6}{x^2} + \alpha_1 x^2 + \alpha_2 x^4 + \alpha_3 x^6 + \alpha_4 x^8 + \cdots \tag{33}$$

This potential satisfies the Novikov equation [30] :

$$\sum_{i=-1}^{2} c_i \frac{\delta S_i}{\delta u} = 0,$$

where δ is the variational operator of the calculus of variations, c_i are constants and

$$S_{-1} = \int u\,dx, \quad S_1 = \int \left(\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + u^3 \right) dx,$$

$$S_0 = \int u^2 dx, \qquad S_2 = \int \left(\frac{1}{2} \left(\frac{\partial^2 u}{\partial x^2} \right)^2 - \frac{5}{2} u^2 \frac{\partial^2 u}{\partial x^2} + \frac{5}{2} u^4 \right) dx,$$

are the first integrals of the Korteweg-de Vries equation. Therefore, the algebraic curve associated with the potential(33)has the form [4]:

$$\begin{aligned} w^2 &= z^5 - 35\alpha_1 z^3 - 63\alpha_2 z^2 + \frac{1}{2}\left(567\alpha_1^2 + 297\alpha_3\right) z \\ &\quad +1377\alpha_1\alpha_2 - 1287\alpha_4, \\ &= \prod_{i=1}^{5} (z - z_i). \end{aligned} \tag{34}$$

It follows from the trace formulae [26] , that

$$s_1 + s_2 = -\sum_{i=1}^{N} \wp(x - x_j) - \frac{C}{2},$$

$$s_1 s_2 = 3 \sum_{\substack{i,j=1 \\ i\langle j}}^{N} \wp(x - x_i)\wp(x - x_j) - \frac{1}{8} N g_2 + \frac{1}{2} \sum_{\substack{i,j=1 \\ i\langle j}}^{5} z_i z_j + \frac{3C}{2} \sum_{i=1}^{N} \wp(x - x_i) + \frac{3C^2}{8},$$

where $z_1, ..., z_5$ are the branching points of the curve (34) .
We want the curve $\Gamma(6)$ to be associated with the elliptic potential curve (34) . Let z_α, z_β be two distinct branch points of the curve (34) which are the shifted values of the branch points λ_1, λ_2 of the curve $\Gamma(6)$. From system (2) , we obtain

$$\ddot{y}_1 = \left(\lambda_1 - y_1^2 - y_2^2\right) y_1,$$

$$\ddot{y}_2 = \left(\lambda_2 - y_1^2 - y_2^2\right) y_2.$$

This together with (10) , (35) and (32) implies that

$$\ddot{y}_1 - \mathcal{U}(x) y_1 = \left(\lambda_1 - 2(z_\alpha + y_\beta)\right) y_1,$$

$$\ddot{y}_2 - \mathcal{U}(x) y_2 = \left(\lambda_2 - 2(z_\alpha + y_\beta)\right) y_2.$$

Consequently, we have

Theorem 4. *Suppose that $\lambda_1 = 3z_\alpha + 2z_\beta$ and $\lambda_2 = 2z_\alpha + 3z_\beta$ where z_α and z_β are two distinct branch points of the curve (34) . Then, formula for elliptic solutions of (2) or (12) is given by*

$$y_1^2 = \frac{1}{z_{\alpha\beta}} \left\{ \Phi + 2z_\alpha^2 + (3C - 2z_\alpha) \sum_{i=1}^{N} \wp(x - x_i) + \left(\frac{3C}{4} - z_\alpha \right) C \right\},$$

$$y_2^2 = \frac{-1}{z_{\alpha\beta}} \left\{ \Phi + 2z_\beta^2 + (3C - 2z_\beta) \sum_{i=1}^{N} \wp(x - x_i) + \left(\frac{3C}{4} - z_\beta \right) C \right\},$$

where $z_{\alpha\beta} \equiv z_\alpha - z_\beta$,

$$\Phi \equiv 6 \sum_{1 \le i\langle j \le N} \wp(x - x_i)\wp(x - x_j) - \frac{N g_2}{4} + \sum_{1 \le i < j \le 5} z_i z_j,$$

and $z_1, ..., z_5$ are the branching points of the curve (34) .

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