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Review

Prym varieties and applications

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ABSTRACT

The classical definition of Prym varieties deals with the unramified covers of curves. The aim of this article is to give explicit algebraic descriptions of the Prym varieties associated with ramified double covers of algebraic curves. We make a careful study of the connection with the concept of algebraic completely integrable systems and we apply the methods to some problems such as the Hénon–Heiles system, the Kowalewski rigid body motion and Kirchhoff's equations of motion of a solid in an ideal fluid.

Contents

1. Introduction

During the last decades, algebraic geometry has become a tool for solving differential equations and spectral questions of mechanics and mathematical physics. This paper consists of two separate but related topics: the first part purely algebraic-geometric, the second one on Prym varieties in algebraic integrability.

Prym variety $\text{Prym}(\mathcal{C}/\mathcal{C}_0)$ is a subabelian variety of the jacobian variety $\text{Jac}(\mathcal{C}) = \text{Pic}^0(\mathcal{C}) = H^1(\mathcal{O}_C)/H^1(\mathcal{C},\mathbb{Z})$ constructed from a double cover $\mathcal{C}$ of a curve $\mathcal{C}_0$: if σ is the involution on C interchanging sheets, then σ extends by

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linearity to a map $\sigma : \mathrm{Jac}(\mathcal{C}) \longrightarrow \mathrm{Jac}(\mathcal{C})$. Up to some points of order two, $\mathrm{Jac}(\mathcal{C})$ splits into an even part and an odd part: the even part is $\mathrm{Jac}(\mathcal{C}_0)$ and the odd part is a $\mathrm{Prym}(\mathcal{C}/\mathcal{C}_0)$. The classical definition of Prym varieties deals with the unramified double covering of curves and was introduced by W. Schottky and H. W. E. Jung in relation with the Schottky problem [6] of characterizing jacobian varieties among all principally polarized abelian varieties (an abelian variety is a complex torus that can be embedded into projective space). The theory of Prym varieties was dormant for a long time, until revived by D. Mumford around 1970. It now plays a substantial role in some contemporary theories, for example integrable systems [1,2,5,8,10,14,15,17], the Kadomtsev–Petviashvili equation (KP equation), in the deformation theory of two-dimensional Schrödinger operators [19], in relation to Calabi–Yau three-folds and string theory, in the study of the generalized theta divisors on the moduli spaces of stable vector bundles over an algebraic curve [7,11],…

Integrable hamiltonian systems are nonlinear ordinary differential equations described by a hamiltonian function and possessing sufficiently many independent constants of motion in involution. By the Arnold–Liouville theorem [4,16], the regular compact level manifolds defined by the intersection of the constants of motion are diffeomorphic to a real torus on which the motion is quasi-periodic as a consequence of the following differential geometric fact; a compact and connected n-dimensional manifold on which there exist n vector fields which commute and are independent at every point is diffeomorphic to an n-dimensional real torus and each vector field will define a linear flow there. A dynamical system is algebraic completely integrable (in the sense of Adler–van Moerbeke [1]) if it can be linearized on a complex algebraic torus $\mathbb{C}^n/\mathit{lattice}$ (=abelian variety). The invariants (often called first integrals or constants) of the motion are polynomials and the phase space coordinates (or some algebraic functions of these) restricted to a complex invariant variety defined by putting these invariants equals to generic constants, are meromorphic functions on an abelian variety. Moreover, in the coordinates of this abelian variety, the flows (run with complex time) generated by the constants of the motion are straight lines.

One of the remarkable developments in recent mathematics is the interplay between algebraic completely integrable systems and Prym varieties. The period of these Prym varieties provide the exact periods of the motion in terms of explicit abelian integrals. The aim of the first part of the present paper is to give explicit algebraic descriptions of the Prym varieties associated to ramified double covers of algebraic curves. The basic algebraic tools are known and can be found in the book by Arbarello, Cornalba, Griffiths, Harris [3] and in Mumford's paper [18]. In the second part of the paper, we make a careful study of the connection with the concept of algebraic completely integrable systems and we apply the methods to some problems such as the Hénon–Heiles system, the Kowalewski rigid body motion and Kirchhoff's equations of motion of a solid in an ideal fluid. The motivation was the excellent Haine's paper [10] on the integration of the Euler–Arnold equations associated to a class of geodesic flow on $SO(4)$. The Kowalewski's top and the Clebsch's case of Kirchhoff's equations describing the motion of a solid body in a perfect fluid were integrated in terms of genus two hyperelliptic functions by Kowalewski [13] and Kötter [12] as a result of complicated and mysterious computations. The concept of algebraic complete integrability (Adler–van Moerbeke) throws a completely new light on these two systems. Namely, in both cases (see [14] for Kowalewski's top and [10] for geodesic flow on $SO(4)$), the affine varieties in $\mathbb{C}^6$ obtained by intersecting the four polynomial invariants of the flow are affine parts of Prym varieties of genus 3 curves which are double cover of elliptic curves. Such Prym varieties are not principally polarized and so they are not isomorphic but only isogenous to Jacobi varieties of genus two hyperelliptic curves. This was a total surprise as it was generically believed that only jacobians would appear as invariants manifolds of such systems. The method that is used for reveiling the Pryms is due to Haine [10]. By now many algebraic completely integrable systems are known to linearize on Prym varieties.

2. Prym varieties

Let $\varphi : \mathcal{C} \longrightarrow \mathcal{C}_0$ be a double covering with $2n$ branch points where $\mathcal{C}$ and $\mathcal{C}_0$ are nonsingular complete curves. Let $\sigma : \mathcal{C} \longrightarrow \mathcal{C}$ be the involution exchanging sheets of $\mathcal{C}$ over $\mathcal{C}_0 = \mathcal{C}/\sigma$. By Hurwitz's formula, the genus of $\mathcal{C}$ is $g = 2g_0 + n - 1$, where g_0 is the genus of $\mathcal{C}_0$. Let $\mathcal{O}_{\mathcal{C}}$ be the sheaf of holomorphic functions on $\mathcal{C}$. Let $S^g(\mathcal{C})$ be the g-th symmetric power of $\mathcal{C}$ (the totality of unordered sets of points of $\mathcal{C}$). On $S^g(\mathcal{C})$, two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ are linearly equivalent (in short, $\mathcal{D}_1 \equiv \mathcal{D}_2$) if their difference $\mathcal{D}_1 - \mathcal{D}_2$ is the divisor of a meromorphic function or equivalently if and only if $\int_{\mathcal{D}_1}^{\mathcal{D}_2} \omega = \int_\gamma \omega, \forall \omega \in \Omega^1_{\mathcal{C}}$, where $\Omega^1_{\mathcal{C}}$ is the sheaf of holomorphic 1-forms on $\mathcal{C}$ and γ is a closed path on $\mathcal{C}$. We define the jacobian (Jacobi variety) of $\mathcal{C}$, to be $\mathrm{Jac}(\mathcal{C}) = S^g(\mathcal{C})/\equiv$. To be precise, the jacobian $\mathrm{Jac}(\mathcal{C}) = \mathrm{Pic}^0(\mathcal{C})$ of $\mathcal{C}$ is the connected component of its Picard group parametrizing degree 0 invertible sheaves. It is a compact commutative algebraic group, i.e., a complex torus. Indeed, from the fundamental exponential sequence, we get an isomorphism

$$\mathrm{Jac}(\mathcal{C}) = \mathrm{Pic}^0(\mathcal{C}) = H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})/H_1(\mathcal{C}, \mathbb{Z}) \simeq H^0(\mathcal{C}, \Omega^1_{\mathcal{C}})^*/H_1(\mathcal{C}, \mathbb{Z}) \simeq \mathbb{C}^g/\mathbb{Z}^{2g},$$

via the Serre's duality. Consider the following mapping

$$S^g(\mathcal{C}) \longrightarrow \mathbb{C}^g/L_\Omega, \qquad \sum_{k=1}^g \mu_k \longmapsto \sum_{k=1}^g \int^{\mu_k(t)} (\omega_1, \ldots, \omega_g) = t(k_1, \ldots, k_g),$$

where $(\omega_1, \ldots, \omega_g)$ is a basis of $\Omega^1_{\mathcal{C}}$, L_Ω is the lattice associated to the period matrix Ω and $\mu_1, \ldots, \mu_g$ some appropriate variables defined on a non empty Zariski open set. Let $(a_1, \ldots, a_{g_0}, \ldots, a_g, b_1, \ldots, b_{g_0}, \ldots, b_g)$, be a canonical homology basis of $H_1(\mathcal{C}, \mathbb{Z})$ such that $\sigma(a_1) = a_{g_0+n}, \ldots, \sigma(a_{g_0}) = a_g, \sigma(a_{g_0+1}) = -a_{g_0+1}, \ldots, \sigma(a_{g_0+n-1}) = -a_{g_0+n-1}, \sigma(b_1) = b_{g_0+n}, \ldots, \sigma(b_{g_0}) = b_g, \sigma(b_{g_0+1}) = -b_{g_0+1}, \ldots, \sigma(b_{g_0+n-1}) = -b_{g_0+n-1}$, for the involution σ. Notice that $\varphi(a_{g_0+1}), \ldots, \varphi(a_{g_0+n-1}), \varphi(b_{g_0+1}), \ldots, \varphi(b_{g_0+n-1})$ are homologous to zero on $\mathcal{C}_0$. Let $(\omega_1, \ldots, \omega_g)$ be a basis of holomorphic differentials on $\mathcal{C}$ where $\omega_{g_0+n}, \ldots, \omega_g$, are holomorphic differentials on $\mathcal{C}_0$ and

$$\sigma^*(\omega_j) = \begin{cases} -\omega_j, & 1 \le j \le g_0 + n - 1, \\ \omega_j, & g_0 + n \le j \le g, \end{cases}$$

the pullback of ω_j. The period matrix Ω of $\mathrm{Jac}(\mathcal{C})$ is explicitly given by

$$\Omega = \begin{pmatrix} A & B & C & D & E & F \\ G & H & I & J & K & L \end{pmatrix},$$

where $A, \ldots, L$ denote the following matrices:

$$A = \begin{pmatrix} \int_{a_1} \omega_1 & \cdots & \int_{a_{g_0}} \omega_1 \\ \vdots & & \vdots \\ \int_{a_1} \omega_{g_0+n-1} & \cdots & \int_{a_{g_0}} \omega_{g_0+n-1} \end{pmatrix}, \tag{1}$$

$$B = \begin{pmatrix} \int_{a_{g_0+1}} \omega_1 & \cdots & \int_{a_{g_0+n-1}} \omega_1 \\ \vdots & & \vdots \\ \int_{a_{g_0+1}} \omega_{g_0+n-1} & \cdots & \int_{a_{g_0+n-1}} \omega_{g_0+n-1} \end{pmatrix},$$

$$C = \begin{pmatrix} \int_{a_{g_0+n}} \omega_1 & \cdots & \int_{a_g} \omega_1 \\ \vdots & & \vdots \\ \int_{a_{g_0+n}} \omega_{g_0+n-1} & \cdots & \int_{a_g} \omega_{g_0+n-1} \end{pmatrix},$$

$$D = \begin{pmatrix} \int_{b_1} \omega_1 & \cdots & \int_{b_{g_0}} \omega_1 \\ \vdots & & \vdots \\ \int_{b_1} \omega_{g_0+n-1} & \cdots & \int_{b_{g_0}} \omega_{g_0+n-1} \end{pmatrix},$$

$$E = \begin{pmatrix} \int_{b_{g_0+1}} \omega_1 & \cdots & \int_{b_{g_0+n-1}} \omega_1 \\ \vdots & & \vdots \\ \int_{b_{g_0+1}} \omega_{g_0+n-1} & \cdots & \int_{b_{g_0+n-1}} \omega_{g_0+n-1} \end{pmatrix},$$

$$F = \begin{pmatrix} \int_{b_{g_0+n}} \omega_1 & \cdots & \int_{b_g} \omega_1 \\ \vdots & & \vdots \\ \int_{b_{g_0+n}} \omega_{g_0+n-1} & \cdots & \int_{b_g} \omega_{g_0+n-1} \end{pmatrix},$$

$$G = \begin{pmatrix} \int_{a_1} \omega_{g_0+n} & \cdots & \int_{a_{g_0}} \omega_{g_0+n} \\ \vdots & & \vdots \\ \int_{a_1} \omega_g & \cdots & \int_{a_{g_0}} \omega_g \end{pmatrix},$$

$$H = \begin{pmatrix} \int_{a_{g_0+1}} \omega_{g_0+n} & \cdots & \int_{a_{g_0+n-1}} \omega_{g_0+n} \\ \vdots & & \vdots \\ \int_{a_{g_0+1}} \omega_g & \cdots & \int_{a_{g_0+n-1}} \omega_g \end{pmatrix},$$

$$I=\begin{pmatrix}\int_{a_{g_0+n}}\omega_{g_0+n} & \cdots & \int_{a_g}\omega_{g_0+n}\\ \vdots & & \vdots\\ \int_{a_{g_0+n}}\omega_g & \cdots & \int_{a_g}\omega_g\end{pmatrix},$$

$$J=\begin{pmatrix}\int_{b_1}\omega_{g_0+n} & \cdots & \int_{b_{g_0}}\omega_{g_0+n}\\ \vdots & & \vdots\\ \int_{b_1}\omega_g & \cdots & \int_{b_{g_0}}\omega_g\end{pmatrix},$$

$$K=\begin{pmatrix}\int_{b_{g_0+1}}\omega_{g_0+n} & \cdots & \int_{b_{g_0+n-1}}\omega_{g_0+n}\\ \vdots & & \vdots\\ \int_{b_{g_0+1}}\omega_g & \cdots & \int_{b_{g_0+n-1}}\omega_g\end{pmatrix},$$

and

$$L=\begin{pmatrix}\int_{b_{g_0+n}}\omega_{g_0+n} & \cdots & \int_{b_g}\omega_{g_0+n}\\ \vdots & & \vdots\\ \int_{b_{g_0+n}}\omega_g & \cdots & \int_{b_g}\omega_g\end{pmatrix}.$$

Notice that

$$\begin{aligned}\int_{a_{g_0+1}}\omega_j &= -\int_{\sigma(a_{g_0+1})}\omega_j,\\ &= -\int_{a_{g_0+1}}\sigma^*(\omega_j),\\ &= \begin{cases}\int_{a_{g_0+1}}\omega_j, & 1\le j\le g_0+n-1,\\ -\int_{a_{g_0+1}}\omega_j, & g_0+n\le j\le g,\end{cases}\\ &\vdots\\ \int_{a_{g_0+n-1}}\omega_j &= \begin{cases}\int_{a_{g_0+n-1}}\omega_j, & 1\le j\le g_0+n-1,\\ -\int_{a_{g_0+n-1}}\omega_j, & g_0+n\le j\le g,\end{cases}\\ \int_{b_{g_0+1}}\omega_j &= -\int_{\sigma\left(b_{g_0+1}\right)}\omega_j,\\ &= -\int_{b_{g_0+1}}\sigma^*\left(\omega_j\right),\\ &= \begin{cases}\int_{b_{g_0+1}}\omega_j, & 1\le j\le g_0+n-1,\\ -\int_{b_{g_0+1}}\omega_j, & g_0+n\le j\le g,\end{cases}\\ &\vdots\\ \int_{b_{g_0+1}}\omega_j &= \begin{cases}\int_{b_{g_0+1}}\omega_j, & 1\le j\le g_0+n-1,\\ -\int_{b_{g_0+1}}\omega_j, & g_0+n\le j\le g,\end{cases}\end{aligned}$$

$$\begin{aligned}\int_{a_{g_0+n}}\omega_j &= \int_{\sigma(a_1)}\omega_j,\\ &= \int_{a_1}\sigma^*(\omega_j),\\ &= \begin{cases}-\int_{a_1}\omega_j, & 1\le j\le g_0+n-1,\\ \int_{a_1}\omega_j, & g_0+n\le j\le g,\end{cases}\end{aligned}$$

$$\vdots$$

$$\int_{a_{g_0+n}}\omega_j = \begin{cases}-\int_{a_{g_0}}\omega_j, & 1\le j\le g_0+n-1,\\ \int_{a_{g_0}}\omega_j, & g_0+n\le j\le g,\end{cases}$$

and

$$\begin{aligned}\int_{b_{g_0+n}}\omega_j &= \int_{\sigma(b_1)}\omega_j,\\ &= \int_{b_1}\sigma^*(\omega_j),\\ &= \begin{cases}-\int_{b_1}\omega_j, & 1\le j\le g_0+n-1,\\ \int_{b_1}\omega_j, & g_0+n\le j\le g,\end{cases}\end{aligned}$$

$$\vdots$$

$$\int_{b_g}\omega_j = \begin{cases}-\int_{b_{g_0}}\omega_j, & 1\le j\le g_0+n-1,\\ \int_{b_{g_0}}\omega_j, & g_0+n\le j\le g.\end{cases}$$

Then, $C=-A,\ F=-D,\ H=O,\ I=G,\ K=O,\ L=J$, where O is the null matrix and therefore

$$\begin{aligned}\Omega &= \begin{pmatrix}A & B & -A & D & E & -D\\ G & O & G & J & O & J\end{pmatrix},\\ &\equiv \begin{pmatrix}C_1 & C_2 & C_3 & C_4 & C_5 & C_6\end{pmatrix}.\end{aligned}$$

By elementary column operation, we obtain the following matrices:

$$\begin{aligned}\Omega_1 &= \begin{pmatrix}C_1 & C_2 & C_1+C_3 & C_4 & C_5 & C_4+C_6\end{pmatrix},\\ &= \begin{pmatrix}A & B & O & D & E & O\\ G & O & 2G & J & O & 2J\end{pmatrix},\\ \Omega_2 &= \begin{pmatrix}C_1 & C_2 & C_1-C_3 & C_4 & C_5 & C_4-C_6\end{pmatrix},\\ &= \begin{pmatrix}A & B & 2A & D & E & 2D\\ G & O & O & J & O & O\end{pmatrix},\\ \Omega_3 &= \begin{pmatrix}C_1-C_3 & C_2 & C_4-C_6 & C_5 & C_1+C_3 & C_4+C_6\end{pmatrix},\\ &= \begin{pmatrix}2A & B & 2D & E & O & O\\ O & O & O & O & 2G & 2J\end{pmatrix},\\ &= \begin{pmatrix}\Gamma & O\\ O & 2\Delta\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}\Delta &= \begin{pmatrix}G & J\end{pmatrix},\\ &= \begin{pmatrix}\int_{a_1}\omega_{g_0+n} & \cdots & \int_{a_{g_0}}\omega_{g_0+n} & \int_{b_1}\omega_{g_0+n} & \cdots & \int_{b_{g_0}}\omega_{g_0+n}\\ \vdots & & \vdots & \vdots & & \vdots\\ \int_{a_1}\omega_g & \cdots & \int_{a_{g_0}}\omega_g & \int_{b_1}\omega_g & \cdots & \int_{b_{g_0}}\omega_g\end{pmatrix},\end{aligned} \tag{2}$$

and

$$\Gamma = \begin{pmatrix} 2A & B & 2D & E \end{pmatrix},$$

$$= \begin{pmatrix} 2\int_{a_1}\omega_1 & \dots & 2\int_{a_1}\omega_{g_0+n-1} \\ \vdots & & \vdots \\ 2\int_{a_{g_0}}\omega_1 & \dots & 2\int_{a_{g_0}}\omega_{g_0+n-1} \\ \int_{a_{g_0+1}}\omega_1 & \dots & \int_{a_{g_0+1}}\omega_{g_0+n-1} \\ \vdots & & \vdots \\ \int_{a_{g_0+n-1}}\omega_1 & \dots & \int_{a_{g_0+n-1}}\omega_{g_0+n-1} \\ 2\int_{b_1}\omega_1 & \dots & 2\int_{b_1}\omega_{g_0+n-1} \\ \vdots & & \vdots \\ 2\int_{b_{g_0}}\omega_1 & \dots & 2\int_{b_{g_0}}\omega_{g_0+n-1} \\ \int_{b_{g_0+1}}\omega_1 & \dots & \int_{b_{g_0+1}}\omega_{g_0+n-1} \\ \vdots & & \vdots \\ \int_{b_{g_0+n-1}}\omega_1 & \dots & \int_{b_{g_0+n-1}}\omega_{g_0+n-1} \end{pmatrix}^{\intercal}. \quad (3)$$

Let

$$L_\Omega = \left\{ \sum_{i=1}^{g} m_i \int_{a_i} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} + n_i \int_{b_i} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} : m_i, n_i \in \mathbb{Z} \right\},$$

be the period lattice associated to Ω. Let us denote also by $L_{\Omega_1}, L_{\Omega_2}, L_{\Omega_3}$ and L_Δ the period lattices associated respectively to Ω_1, Ω_2, Ω_3 and Δ. Since $L_\Omega = L_{\Omega_1} = L_{\Omega_2} = L_{\Omega_3}$, it follows that the maps

$$\mathbb{C}^{g_0+n-1}/L_\Gamma : \begin{pmatrix} t_1 \\ \vdots \\ t_{g_0+n-1} \end{pmatrix} \bmod L_\Gamma \hookrightarrow \mathbb{C}^g/L_\Omega : \begin{pmatrix} t_1 \\ \vdots \\ t_{g_0+n-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \bmod L_\Omega,$$

$$\mathbb{C}^{g_0}/2L_\Delta : \begin{pmatrix} t_1 \\ \vdots \\ t_{g_0} \end{pmatrix} \bmod 2L_\Delta \hookrightarrow \mathbb{C}^g/L_\Omega : \begin{pmatrix} 0 \\ \vdots \\ 0 \\ t_1 \\ \vdots \\ t_{g_0} \end{pmatrix} \bmod L_\Omega,$$

are injectives. Therefore, the tori $\mathbb{C}^{g_0+n-1}/L_\Gamma$ and $\mathbb{C}^{g_0}/2L_\Delta$ can be embedded into $\mathbb{C}^g/L_\Omega$ and the map

$$\mathbb{C}^g/L_{\Omega_3} = \mathbb{C}^{g_0+n-1}/L_\Gamma \oplus \mathbb{C}^{g_0}/2L_\Delta \longrightarrow \mathbb{C}^g/L_\Omega,$$
$$\times \begin{pmatrix} t_1 \\ \vdots \\ t_g \end{pmatrix} \bmod L_{\Omega_3} \mapsto \begin{pmatrix} t_1 \\ \vdots \\ t_g \end{pmatrix} \bmod L_\Omega,$$

shows that the jacobian variety Jac $(\mathbb{C}_0)$ intersects the torus $\mathbb{C}^{g_0+n-1}/L_\Gamma$ in 2^{2g_0} points. We have then the following diagram

$$\begin{array}{ccccccc}
\mathcal{C} & & Ker\, N_\varphi & & & & \\
\downarrow \sigma & & \downarrow & & & & \\
\mathcal{C} & \xrightarrow{u} & \mathrm{Jac}(\mathcal{C}) & \xleftarrow{u^*} & \mathrm{Jac}^*(\mathcal{C}) & & \\
\downarrow \varphi & & \downarrow N_\varphi & & \uparrow N_\varphi^* & & \\
\mathcal{C}_0 & \xrightarrow{u_0} & \mathrm{Jac}(\mathcal{C}_0) & \xleftarrow{u_0^*} & \mathrm{Jac}^*(\mathcal{C}_0) & & \\
 & & \downarrow \varphi^* & & & & \\
 & & \mathrm{Jac}(\mathcal{C}) & \subset & H^1(\mathcal{C}, \mathcal{O}_\mathcal{C}^*) & \simeq & H^1(\mathcal{C}_0, (\varphi_* \mathcal{O}_\mathcal{C})^*) \\
 & & \downarrow \sigma & & & & \downarrow N_\varphi \\
 & & \mathrm{Jac}(\mathcal{C}) & \xrightarrow{N_\varphi} & \mathrm{Jac}(\mathcal{C}_0) & \hookrightarrow & H^1(\mathcal{C}_0, \mathcal{O}_{\mathcal{C}_0}^*),
\end{array}$$

where $u : z \mapsto$ divisor class$(z - p), p \in \mathcal{C}$, fixed, $u_0 : z_0 \mapsto$ divisor class$(z_0 - p_0), p_0 \in \mathcal{C}_0$, fixed with $p_0 = \varphi(p)$, $N_\varphi : \mathrm{Jac}(\mathcal{C}) \to \mathrm{Jac}(\mathcal{C}_0), \sum m_i p_i \mapsto \sum m_i \varphi(p_i)$ is the norm mapping and $\mathrm{Jac}^*(\mathcal{C}) =$ dual of $\mathrm{Jac}(\mathcal{C})$. The norm map N_φ is surjective. Moreover, the dual $\mathrm{Jac}^*(\mathcal{C})$ of $\mathrm{Jac}(\mathcal{C})$ is isomorphic to $\mathrm{Jac}(\mathcal{C})$. The Prym variety denoted $\mathrm{Prym}(\mathcal{C}/\mathcal{C}_0)$ is defined by

$$\mathrm{Prym}(\mathcal{C}/\mathcal{C}_0) = \left(H^0\left(\mathcal{C}, \Omega^1_\mathcal{C}\right)^-\right)^* / H_1(\mathcal{C}, \mathbb{Z})^-,$$

where $-$ denote the -1 eigenspace for a vector space on which the involution σ acts. Let $D \in \mathrm{Prym}(\mathcal{C}/\mathcal{C}_0)$, i.e.,

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 2\int_0^D \omega_{g_0+n} \\ \vdots \\ 2\int_0^D \omega_g \end{pmatrix} \in L_\Omega \iff \begin{pmatrix} \int_0^D \omega_{g_0+n} \\ \vdots \\ \int_0^D \omega_g \end{pmatrix} \in L_\Delta,$$

$$\iff \begin{pmatrix} \int_0^D \omega_1 \\ \vdots \\ \int_0^D \omega_{g_0+n-1} \\ \int_0^D \omega_{g_0+n} \\ \vdots \\ \int_0^D \omega_g \end{pmatrix} = \begin{pmatrix} t_1 \\ \vdots \\ t_{g_0+n-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Consequently, the Abel–Jacobi map

$$\mathrm{Jac}(\mathcal{C}) \longrightarrow \mathcal{C}^g/L_\Omega, \qquad D \longmapsto \begin{pmatrix} \int_0^D \omega_1 \\ \vdots \\ \int_0^D \omega_{g_0+n-1} \\ \int_0^D \omega_{g_0+n} \\ \vdots \\ \int_0^D \omega_g \end{pmatrix},$$

maps $\mathrm{Prym}(\mathcal{C}/\mathcal{C}_0)$ bi-analtically onto $\mathcal{C}^{g_0+n-1}/L_\Gamma$. By Chow's theorem [9], $\mathrm{Prym}(\mathcal{C}/\mathcal{C}_0)$ is thus a subabelian variety of Jac $(\mathcal{C})$. More precisely, we have

$$\begin{aligned}
\mathrm{Prym}(\mathcal{C}/\mathcal{C}_0) &= (Ker\, N_\varphi)^0, \\
&= Ker(1_{\mathrm{Jac}(\mathcal{C})} + \sigma)^0, \\
&= \mathrm{Im}(1_{\mathrm{Jac}(\mathcal{C})} - \sigma) \subset \mathrm{Jac}(\mathcal{C}).
\end{aligned}$$

Equivalently, the involution σ induces an involution

$$\sigma : \mathrm{Jac}(\mathcal{C}) \longrightarrow \mathrm{Jac}(\mathcal{C}), \qquad \text{class of } D \longmapsto \text{class of } \sigma D,$$

and up to some points of order two, $\mathrm{Jac}(\mathcal{C})$ splits into an even part $\mathrm{Jac}\,(\mathcal{C}_0)$ and an odd part $\mathrm{Prym}(\mathcal{C}/\mathcal{C}_0)$:

$$\mathrm{Jac}(\mathcal{C}) = \mathrm{Prym}(\mathcal{C}/\mathcal{C}_0) \oplus \mathrm{Jac}\,(C_0)\,,$$

with

$$\begin{aligned}
&\dim \mathrm{Jac}(C_0) = g_0,\\
&\dim \mathrm{Jac}(\mathcal{C}) = g = 2g_0 + n - 1,\\
&\dim \mathrm{Prym}(\mathcal{C}/\mathcal{C}_0) = g - g_0 = g_0 + n - 1.
\end{aligned}$$

Observe that $\Delta\,(2)$ (resp. $\Gamma\,(3)$) is the period matrix of $\mathrm{Jac}\,(\mathcal{C}_0)$ (resp. $\mathrm{Prym}(\mathcal{C}/\mathcal{C}_0)$). Write $\Gamma = (U, V)$, with

$$U = \begin{pmatrix} 2\int_{a_1}\omega_1 & \dots & 2\int_{a_{g_0}}\omega_1 & \int_{a_{g_0+1}}\omega_1 & \dots & \int_{a_{g_0+n-1}}\omega_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ 2\int_{a_1}\omega_{g_0+n-1} & \dots & 2\int_{a_{g_0}}\omega_{g_0+n-1} & \int_{a_{g_0+1}}\omega_{g_0+n-1} & \dots & \int_{a_{g_0+n-1}}\omega_{g_0+n-1} \end{pmatrix},$$

$$V = \begin{pmatrix} 2\int_{b_1}\omega_1 & \dots & 2\int_{b_{g_0}}\omega_1 & \int_{b_{g_0+1}}\omega_1 & \dots & \int_{b_{g_0+n-1}}\omega_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ 2\int_{b_1}\omega_{g_0+n-1} & \dots & 2\int_{b_{g_0}}\omega_{g_0+n-1} & \int_{b_{g_0+1}}\omega_{g_0+n-1} & \dots & \int_{b_{g_0+n-1}}\omega_{g_0+n-1} \end{pmatrix},$$

and let us call

$$e_1 = \begin{pmatrix} 2\int_{a_1}\omega_1 \\ \vdots \\ 2\int_{a_1}\omega_{g_0+n-1} \end{pmatrix}, \dots, e_{g_0} = \begin{pmatrix} 2\int_{a_{g_0}}\omega_1 \\ \vdots \\ 2\int_{a_{g_0}}\omega_{g_0+n-1} \end{pmatrix},$$

$$e_{g_0+1} = \begin{pmatrix} \int_{a_{g_0+1}}\omega_1 \\ \vdots \\ \int_{a_{g_0+1}} \end{pmatrix}, \dots, e_{g_0+n-1} = \begin{pmatrix} \int_{a_{g_0+n-1}}\omega_1 \\ \vdots \\ \int_{a_{g_0+n-1}}\omega_{g_0+n-1} \end{pmatrix}.$$

Then, in the new basis $(\lambda_1, \dots, \lambda_{g_0+n-1})$ where

$$\lambda_j = \frac{e_j}{\delta_j}, \quad \delta_j = \begin{cases} 1 & \text{pour } 1 \le j \le g_0, \\ 2 & \text{pour } g_0 + 1 \le j \le g_0 + n - 1, \end{cases}$$

the period matrix Γ takes the canonical form (Δ_δ, Z), with $\Delta_\delta = \mathrm{diag}(\delta_1, \dots, \delta_n)$ and $Z = \Delta_\delta U^{-1} V$, symmetric and $\mathrm{Im} > 0$. Then

$$\begin{aligned}
\Gamma^* &= \left(\delta_{g_0+n-1}\Delta_\delta^{-1}, \delta_{g_0+n-1}\Delta_\delta^{-1} Z \Delta_\delta^{-1}\right),\\
&= (\delta_{g_0+n-1}\Delta_\delta^{-1}, \delta_{g_0+n-1} U^{-1} V \Delta_\delta^{-1}),\\
&= \left(\delta_{g_0+n-1}\Delta_\delta^{-1}, \delta_{g_0+n-1}\Delta_\delta^{-1}\left(U^*\right)^{-1} V^*\right),
\end{aligned}$$

and

$$\Gamma^* = (U^*, V^*) = (A\;B\;D\;E),$$

is the period matrix of the dual abelian variety $\mathrm{Prym}^*(\mathcal{C}/\mathcal{C}_0)$. The above discussion is summed up in the following statement:

Theorem 1. *Let $\varphi : \mathcal{C} \longrightarrow \mathcal{C}_0$ be a double covering where $\mathcal{C}$ and $\mathcal{C}_0$ are nonsingular algebraic curves with jacobians $\mathrm{Jac}(\mathcal{C})$ and $\mathrm{Jac}(\mathcal{C}_0)$. Let $\sigma : \mathcal{C} \longrightarrow \mathcal{C}$ be the involution exchanging sheets of $\mathcal{C}$ over $\mathcal{C}_0 = \mathcal{C}/\sigma$. This involution extends by linearity to a map (which will again be denoted by σ) $\sigma : \mathrm{Jac}(\mathcal{C}) \longrightarrow \mathrm{Jac}(\mathcal{C})$ and up some points of order two, $\mathrm{Jac}(\mathcal{C})$ splits into an even part,i.e., $\mathrm{Jac}(\mathcal{C}_0)$ and an odd part (called Prym variety) denoted $\mathrm{Prym}(\mathcal{C}/\mathcal{C}_0)$ and defined by*

$$\mathrm{Prym}(\mathcal{C}/\mathcal{C}_0) = \left(H^0\left(\mathcal{C}, \Omega^1_{\mathcal{C}}\right)^-\right)^* / H_1\left(\mathcal{C}, \mathbb{Z}\right)^-,$$

where $\Omega^1_{\mathcal{C}}$ is the sheaf of holomorphic 1-forms on $\mathcal{C}$ and $-$ denote the -1 eigenspace for a vector space on which j acts. To be precise, we have

$$\mathrm{Jac}(\mathcal{C}) = \mathrm{Prym}(\mathcal{C}/\mathcal{C}_0) \oplus \mathrm{Jac}\,(\mathcal{C}_0)\,,$$

with $\dim \mathrm{Jac}\,(\mathcal{C}_0) =$ *genus* g_0 *of* $\mathcal{C}_0$, $\dim \mathrm{Jac}\,(\mathcal{C}) =$ *genus* g *of* $\mathcal{C} = 2g_0 + n - 1$ *and* $\dim \mathrm{Prym}(\mathcal{C}/\mathcal{C}_0) = g - g_0 = g_0 + n + 1$, *with* $2n$ *branch points. Moreover, if*

$$\Omega = \begin{pmatrix} A & B & C & D & E & F \\ G & H & I & J & K & L \end{pmatrix},$$

is the period matrix of $\mathrm{Jac}(\mathcal{C})$ *where* $A, \ldots, L$ *denote the matrices* (1)*, then the period matrices of* $\mathrm{Jac}\,(\mathcal{C}_0)$, $\mathrm{Prym}\,(\mathcal{C}/\mathcal{C}_0)$ *and the dual* $\mathrm{Prym}^*\,(\mathcal{C}/\mathcal{C}_0)$ *of* $\mathrm{Prym}\,(\mathcal{C}/\mathcal{C}_0)$, *are respectively* $\Delta = (G\,H)$, $\Gamma = (2A\,B\,2D\,E)$ *and* $\Gamma^* = (A\,B\,D\,E)$.

3. Algebraic complete integrability

3.1. A survey on abelian varieties and algebraic integrability

We give some basic facts about integrable hamiltonian systems and some results about abelian surfaces which will be used, as well as the basic techniques to study two-dimensional algebraic completely integrable systems. Let $T = \mathbb{C}^n/\Lambda$ be a n-dimensional abelian variety where Λ is the lattice generated by the $2n$ columns $\lambda_1, \ldots, \lambda_{2n}$ of the $n \times 2n$ period matrix Ω and let D be a divisor on T. Define $\mathcal{L}(\mathcal{D}) = \{f \text{ meromorphic on } T : (f) \geq -\mathcal{D}\}$, i.e., for $\mathcal{D} = \sum k_j \mathcal{D}_j$ a function $f \in \mathcal{L}(\mathcal{D})$ has at worst a k_j-fold pole along $\mathcal{D}_j$. The divisor $\mathcal{D}$ is called ample when a basis $(f_0, \ldots, f_N)$ of $\mathcal{L}(k\mathcal{D})$ embeds T smoothly into $\mathbb{P}^N$ for some k, via the map $T \to \mathbb{P}^N$, $p \mapsto [1 : f_1(p) : \ldots : f_N(p)]$, then $k\mathcal{D}$ is called very ample. It is known that every positive divisor $\mathcal{D}$ on an irreducible abelian variety is ample and thus some multiple of $\mathcal{D}$ embeds M into $\mathbb{P}^N$. By a theorem of Lefschetz, any $k \geq 3$ will work. Moreover, there exists a complex basis of $\mathbb{C}^n$ such that the lattice expressed in that basis is generated by the columns of the $n \times 2n$ period matrix

$$\left(\begin{array}{ccc|c} \delta_1 & & 0 & \\ & \ddots & & Z \\ 0 & & \delta_n & \end{array}\right),$$

with $Z^\top = Z$, $\mathrm{Im}\,Z > 0$, $\delta_j \in \mathbb{N}^*$ and $\delta_j | \delta_{j+1}$. The integers δ_j which provide the so-called polarization of the abelian variety T are then related to the divisor as follows:

$$\dim \mathcal{L}(\mathcal{D}) = \delta_1 \ldots \delta_n. \tag{4}$$

In the case of a 2-dimensional abelian varieties (surfaces), even more can be stated: the geometric genus g of a positive divisor $\mathcal{D}$ (containing possibly one or several curves) on a surface T is given by the adjunction formula

$$g(\mathcal{D}) = \frac{K_T.\mathcal{D} + \mathcal{D}.\mathcal{D}}{2} + 1, \tag{5}$$

where K_T is the canonical divisor on T, i.e., the zero-locus of a holomorphic 2-form, $\mathcal{D}.\mathcal{D}$ denote the number of intersection points of $\mathcal{D}$ with $a + \mathcal{D}$ (where $a + \mathcal{D}$ is a small translation by a of $\mathcal{D}$ on T), where as the Riemann–Roch theorem for line bundles on a surface tells you that

$$\chi(\mathcal{D}) = p_a(T) + 1 + \frac{1}{2}(\mathcal{D}.\mathcal{D} - \mathcal{D}K_T), \tag{6}$$

where $p_a\,(T)$ is the arithmetic genus of T and $\chi(\mathcal{D})$ the Euler characteristic of $\mathcal{D}$. To study abelian surfaces using Riemann surfaces on these surfaces, we recall that

$$\begin{aligned} \chi(\mathcal{D}) &= \dim H^0(T, \mathcal{O}_T(\mathcal{D})) - \dim H^1(T, \mathcal{O}_T(D)), \\ &= \dim \mathcal{L}(\mathcal{D}) - \dim H^1(T, \Omega^2(\mathcal{D} \otimes K_T^*)), \text{ (Kodaira–Serre duality)}, \\ &= \dim \mathcal{L}(\mathcal{D}), \text{ (Kodaira vanishing theorem)}, \end{aligned} \tag{7}$$

whenever $\mathcal{D} \otimes K_T^*$ defines a positive line bundle. However for abelian surfaces, K_T is trivial and $p_a(T) = -1$; therefore combining relations (4)–(7), we obtain

$$\chi\,(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}) = \frac{\mathcal{D}.\mathcal{D}}{2} = g\,(\mathcal{D}) - 1 = \delta_1\delta_2. \tag{8}$$

Let M be a $2n$-dimensional differentiable manifold and ω a closed non-degenerate differential 2-form. The pair (M, ω) is called a symplectic manifold. Let $H : M \to \mathbb{R}$ be a smooth function. A hamiltonian system on (M, ω) with hamiltonian H can be written in the form

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \ldots, \dot{q}_n = \frac{\partial H}{\partial p_n}, \qquad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \ldots, \dot{p}_n = -\frac{\partial H}{\partial q_n}, \tag{9}$$

where $(q_1, \dots, q_n, p_1, \dots, p_n)$ are coordinates in M. Thus the hamiltonian vector field X_H is defined by $X_H = \sum_{k=1}^{n} \left(\frac{\partial H}{\partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial}{\partial p_k} \right)$. If F is a smooth function on the manifold M, the Poisson bracket $\{F, H\}$ of F and H is defined by

$$X_H F = \sum_{k=1}^{n} \left(\frac{\partial H}{\partial p_k} \frac{\partial F}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial F}{\partial p_k} \right) = \{F, H\}. \tag{10}$$

A function F is an invariant (first integral) of the hamiltonian system (9) if and only if the Lie derivative of F with respect X_H is identically zero. The functions F and H are said to be in involution or to commute, if $\{F, H\} = 0$. Note that Eqs. (9) and (10) can be written in more compact form

$$\dot{x} = J \frac{\partial H}{\partial x}, \quad x = (q_1, \dots, q_n, p_1, \dots, p_n)^\top,$$
$$\{F, H\} = \left\langle \frac{\partial F}{\partial x}, J \frac{\partial H}{\partial x} \right\rangle = \sum_{k,l=1}^{n} J_{kl} \frac{\partial F}{\partial x_k} \frac{\partial H}{\partial x_l},$$

with $J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}$, a skew-symmetric matrix where I is the $n \times n$ unit matrix and O the $n \times n$ zero matrix. A hamiltonian system is completely integrable in the sense of Liouville if there exist n invariants $H_1 = H, H_2, \dots, H_n$ in involution (i.e., such that the associated Poisson bracket $\{H_k, H_l\}$ all vanish) with linearly independent gradients (i.e., $dH_1 \wedge \dots \wedge dH_n \neq 0$). For generic $c = (c_1, \dots, c_n)$ the level set $M_c = \{H_1 = c_1, \dots, H_n = c_n\}$ will be an n-manifold, and since $X_{H_k} H_l = \{H_k, H_l\} = 0$, the integral curves of each X_{H_k} will lie in M_c and the vector fields X_{H_k} span the tangent space of M_c. By a theorem of Arnold [4,15], if M_c is compact and connected, it is diffeomorphic to an n-dimensional real torus and each vector field will define a linear flow there. To be precise, in some open neighbourhood of the torus one can introduce regular symplectic coordinates $s_1, \dots, s_n, \varphi_1, \dots, \varphi_n$ in which ω takes the canonical form $\omega = \sum_{k=1}^{n} ds_k \wedge d\varphi_k$. Here the functions s_k (called action-variables) give coordinates in the direction transverse to the torus and can be expressed functionally in terms of the first integrals H_k. The functions φ_k (called angle-variables) give standard angular coordinates on the torus, and every vector field X_{H_k} can be written in the form $\dot{\varphi}_k = h_k(s_1, \dots, s_n)$, that is, its integral trajectories define a conditionally-periodic motion on the torus. Consequently, in a neighbourhood of the torus the hamiltonian vector field X_{H_k} take the following form $\dot{s_k} = 0$, $\dot{\varphi}_k = h_k(s_1, \dots, s_n)$, and can be solved by quadratures.

Consider now hamiltonian problems of the form

$$X_H : \dot{x} = J \frac{\partial H}{\partial x} \equiv f(x), \quad x \in \mathbb{R}^m, \tag{11}$$

where H is the hamiltonian and $J = J(x)$ is a skew-symmetric matrix with polynomial entries in x, for which the corresponding Poisson bracket $\{H_i, H_j\} = \langle \frac{\partial H_i}{\partial x}, J \frac{\partial H_j}{\partial x} \rangle$, satisfies the Jacobi identities. The system (11) with polynomial right hand side will be called algebraically completely integrable (a.c.i.) in the sense of Adler–van Moerbeke [1] when:

(i) The system possesses $n + k$ independent polynomial invariants $H_1, \dots, H_{n+k}$ of which k lead to zero vector fields $J \frac{\partial H_{n+i}}{\partial x}(x) = 0$, $1 \leq i \leq k$, the n remaining ones are in involution (i.e., $\{H_i, H_j\} = 0$) and $m = 2n + k$. For most values of $c_i \in \mathbb{R}$, the invariant varieties $\bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m : H_i = c_i\}$ are assumed compact and connected. Then, according to the Arnold–Liouville theorem, there exists a diffeomorphism

$$\bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m : H_i = c_i\} \to \mathbb{R}^n/\text{Lattice},$$

and the solutions of the system (11) are straight lines motions on these tori.

(ii) The invariant varieties, thought of as affine varieties

$$\mathcal{A} = \bigcap_{i=1}^{n+k} \{H_i = c_i, x \in \mathbb{C}^m\},$$

in $\mathbb{C}^m$ can be completed into complex algebraic tori, i.e., $\mathcal{A} \cup \mathcal{D} = \mathbb{C}^n/\text{Lattice}$, where $\mathbb{C}^n/\text{Lattice}$ is a complex algebraic torus (i.e., abelian variety) and $\mathcal{D}$ a divisor.

Remark 3.1. Algebraic means that the torus can be defined as an intersection $\bigcap_{i=1}^{M} \{P_i(X_0, \dots, X_N) = 0\}$ involving a large number of homogeneous polynomials P_i. In the natural coordinates $(t_1, \dots, t_n)$ of $\mathbb{C}^n/\text{Lattice}$ coming from $\mathbb{C}^n$, the functions $x_i = x_i(t_1, \dots, t_n)$ are meromorphic and (11) defines straight line motion on $\mathbb{C}^n/\text{Lattice}$. Condition (i) means, in particular, there is an algebraic map $(x_1(t), \dots, x_m(t)) \mapsto (\mu_1(t), \dots, \mu_n(t))$ making the following sums linear in t:

$$\sum_{i=1}^{n} \int_{\mu_i(0)}^{\mu_i(t)} \omega_j = d_j t, \quad 1 \leq j \leq n, \ d_j \in \mathbb{C},$$

where $\omega_1, \dots, \omega_n$ denote holomorphic differentials on some algebraic curves.

Adler and van Moerbeke [1] have shown that the existence of a coherent set of Laurent solutions:

$$x_i = \sum_{j=0}^{\infty} x_i^{(j)} t^{j-k_i}, \quad k_i \in \mathbb{Z}, \text{ some } k_i > 0, \tag{12}$$

depending on dim (phase space) $- 1 = m - 1$ free parameters is necessary and sufficient for a hamiltonian system with the right number of constants of motion to be a.c.i. So, if the hamiltonian flow (11) is a.c.i., it means that the variables x_i are meromorphic on the torus $\mathbb{C}^n/\text{Lattice}$ and by compactness they must blow up along a codimension one subvariety (a divisor) $\mathcal{D} \subset \mathbb{C}^n/\text{Lattice}$. By the a.c.i. definition, the flow (11) is a straight line motion in $\mathbb{C}^n/\text{Lattice}$ and thus it must hit the divisor $\mathcal{D}$ in at least one place. Moreover through every point of $\mathcal{D}$, there is a straight line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equations must admit Laurent expansions which depend on the $n-1$ parameters defining $\mathcal{D}$ and the $n+k$ constants c_i defining the torus $\mathbb{C}^n/\text{Lattice}$, the total count is therefore $m - 1 = \dim$ (phase space) $- 1$ parameters.

Assume now hamiltonian flows to be (weight)-homogeneous with a weight $\nu_i \in \mathbb{N}$, going with each variable x_i, i.e.,

$$f_i\left(\alpha^{\nu_1} x_1, \ldots, \alpha^{\nu_m} x_m\right) = \alpha^{\nu_i+1} f_i\left(x_1, \ldots, x_m\right), \quad \forall \alpha \in \mathbb{C}.$$

Observe that then the constants of the motion H can be chosen to be (weight)-homogeneous:

$$H\left(\alpha^{\nu_1} x_1, \ldots, \alpha^{\nu_m} x_m\right) = \alpha^{k} H\left(x_1, \ldots, x_m\right), \quad k \in \mathbb{Z}.$$

If the flow is algebraically completely integrable, the differential equations (11) must admits Laurent series solutions (12) depending on $m-1$ free parameters. We must have $k_i = \nu_i$ and coefficients in the series must satisfy at the 0th step non-linear equations,

$$f_i\left(x_1^{(0)}, \ldots, x_m^{(0)}\right) + g_i x_i^{(0)} = 0, \quad 1 \le i \le m, \tag{13}$$

and at the kth step, linear systems of equations:

$$(L - kI)\, z^{(k)} = \begin{cases} 0 \quad \text{for } k = 1 \\ \text{some polynomial in} \quad x^{(1)}, \ldots, x^{(k-1)} \text{ for } k > 1, \end{cases} \tag{14}$$

where

$$L = \text{Jacobian map of (13)} = \frac{\partial f}{\partial z} + gI\,|_{z=z^{(0)}} .$$

If $m-1$ free parameters are to appear in the Laurent series, they must either come from the non-linear equations (13) or from the eigenvalue problem (14), i.e., L must have at least $m-1$ integer eigenvalues. These are much less conditions than expected, because of the fact that the homogeneity k of the constant H must be an eigenvalue of L. Moreover the formal series solutions are convergent as a consequence of the majorant method [1]. So, the question is how does one prove directly that the system is effectively a.c.i. with abelian space coordinates? The idea of the direct proof used by Adler and van Moerbeke is closely related to the geometric spirit of the (real) Arnold–Liouville theorem discuted above. Namely, a compact complex n-dimensional variety on which there exist n holomorphic commuting vector fields which are independent at every point is analytically isomorphic to a n-dimensional complex torus $\mathbb{C}^n/\text{Lattice}$ and the complex flows generated by the vector fields are straight lines on this complex torus. Now, the affine variety $\mathcal{A}$ is not compact and the main problem will be to complete $\mathcal{A}$ into a non singular compact complex algebraic variety $\widetilde{\mathcal{A}} = \mathcal{A} \cup \mathcal{D}$ in such a way that the vector fields $X_{H_1}, \ldots, X_{H_m}$ extend holomorphically along the divisor $\mathcal{D}$ and remain independent there. If this is possible, $\widetilde{\mathcal{A}}$ is an algebraic complex torus, i.e., an abelian variety and the coordinates x_i restricted to $\mathcal{A}$ are abelian functions. A naive guess would be to take the natural compactification $\overline{\mathcal{A}}$ of $\mathcal{A}$ in $\mathbb{P}^m(\mathbb{C})$. Indeed, this can never work for a general reason: An abelian variety $\widetilde{\mathcal{A}}$ of dimension bigger or equal than two is never a complete intersection, that is it can never be descriped in some projective space $\mathbb{P}^m(\mathbb{C})$ by m-dim $\widetilde{\mathcal{A}}$ global polynomial homogeneous equations. In other words, if $\mathcal{A}$ is to be the affine part of an abelian variety, $\overline{\mathcal{A}}$ must have a singularity somewhere along the locus at infinity $\overline{\mathcal{A}} \cap \{X_0 = 0\}$. In fact, Adler and van Moerbeke [1] showed that the existence of meromorphic solutions to the differential equations (2) depending on $m-1$ free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor. An exposition of such methods and their applications can be found in [1,2].

3.2. The Hénon–Heiles system

The Hénon–Heiles system

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \qquad q_2 = \frac{\partial H}{\partial p_2}, \qquad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \qquad p_2 = -\frac{\partial H}{\partial q_2}, \tag{15}$$

with

$$H \equiv H_1 = \frac{1}{2}\left(p_1^2 + p_2^2 + Aq_1^2 + Bq_2^2\right) + q_1^2 q_2 + 6q_2^3,$$

has another constant of motion

$$H_2 = q_1^4 + 4q_1^2 q_2^2 - 4p_1 (p_1 q_2 - p_2 q_1) + 4Aq_1^2 q_2 + (4A - B)\left(p_1^2 + Aq_1^2\right),$$

where A, B are constant parameters and q_1, q_2, p_1, p_2 are canonical coordinates and momenta, respectively. First studied as a mathematical model to describe the chaotic motion of a test star in an axisymmetric galactic mean gravitational field this system is widely explored in other branches of physics. It well-known from applications in stellar dynamics, statistical mechanics and quantum mechanics. It provides a model for the oscillations of atoms in a three-atomic molecule. The system (15) possesses Laurent series solutions depending on 3 free parameters α, β, γ, namely

$$q_1 = \frac{\alpha}{t} + \left(\frac{\alpha^3}{12} + \frac{\alpha A}{2} - \frac{\alpha B}{12}\right) t + \beta t^2 + q_1^{(4)} t^3 + q_1^{(5)} t^4 + q_1^{(6)} t^5 + \cdots,$$

$$q_2 = -\frac{1}{t^2} + \frac{\alpha^2}{12} - \frac{B}{12} + \left(\frac{\alpha^4}{48} + \frac{\alpha^2 A}{10} - \frac{\alpha^2 B}{60} - \frac{B^2}{240}\right) t^2 + \frac{\alpha\beta}{3} t^3 + \gamma t^4 + \cdots,$$

where $p_1 = \dot{q}_1,\ p_2 = \dot{q}_2$ and

$$q_1^{(4)} = \frac{\alpha AB}{24} - \frac{\alpha^5}{72} + \frac{11\alpha^3 B}{720} - \frac{11\alpha^3 A}{120} - \frac{\alpha B^2}{720} - \frac{\alpha A^2}{8},$$

$$q_1^{(5)} = -\frac{\beta\alpha^2}{12} + \frac{\beta B}{60} - \frac{A\beta}{10},$$

$$q_1^{(6)} = -\frac{\alpha\gamma}{9} - \frac{\alpha^7}{15552} - \frac{\alpha^5 A}{2160} + \frac{\alpha^5 B}{12960} + \frac{\alpha^3 B^2}{25920} + \frac{\alpha^3 A^2}{1440} - \frac{\alpha^3 AB}{4320} + \frac{\alpha AB^2}{1440} - \frac{\alpha B^3}{19440} - \frac{\alpha A^2 B}{288} + \frac{\alpha A^3}{144}.$$

Let $\mathcal{D}$ be the pole solutions restricted to the surface

$$M_c = \bigcap_{i=1}^{2} \left\{x \equiv (q_1, q_2, p_1, p_2) \in \mathbb{C}^4, H_i(x) = c_i\right\},$$

to be precise $\mathcal{D}$ is the closure of the continuous components of the set of Laurent series solutions $x(t)$ such that $H_i(x(t)) = c_i,\ 1 \le i \le 2$, i.e., $\mathcal{D} = t^0 -$ coefficient of M_c. Thus we find an algebraic curve defined by

$$\mathcal{D} : \beta^2 = P_8(\alpha), \tag{16}$$

where

$$P_8(\alpha) = -\frac{7}{15552}\alpha^8 - \frac{1}{432}\left(5A - \frac{13}{18}B\right)\alpha^6 - \frac{1}{36}\left(\frac{671}{15120}B^2 + \frac{17}{7}A^2 - \frac{943}{1260}BA\right)\alpha^4 - \frac{1}{36}\left(4A^3 - \frac{1}{2520}B^3 - \frac{13}{6}A^2 B + \frac{2}{9}AB^2 - \frac{10}{7}c_1\right)\alpha^2 + \frac{1}{36}c_2.$$

The curve $\mathcal{D}$ determined by an eight-order equation is smooth, hyperelliptic and its genus is 3. Moreover, the map

$$\sigma : \mathcal{D} \longrightarrow \mathcal{D}, \qquad (\beta, \alpha) \longmapsto (\beta, -\alpha), \tag{17}$$

is an involution on $\mathcal{D}$ and the quotient $\mathcal{E} = \mathcal{D}/\sigma$ is an elliptic curve defined by

$$\mathcal{E} : \beta^2 = P_4(\zeta), \tag{18}$$

where $P_4(\zeta)$ is the degree 4 polynomial in $\zeta = \alpha^2$ obtained from (16). The hyperelliptic curve $\mathcal{D}$ is thus a 2-sheeted ramified covering of the elliptic curve $\mathcal{E}$ (18),

$$\rho : \mathcal{D} \longrightarrow \mathcal{E}, \qquad (\beta, \alpha) \longmapsto (\beta, \zeta), \tag{19}$$

ramified at the four points covering $\zeta = 0$ and ∞. Following the methods in Adler–van Moerbeke [1,2], the affine surface M_c completes into an abelian surface $\widetilde{M}_c$, by adjoining the divisor $\mathcal{D}$. The latter defines on $\widetilde{M}_c$ a polarization $(1, 2)$. The divisor $2D$ is very ample and the functions $1,\ y_1,\ y_1^2,\ y_2,\ x_1,\ x_1^2 + y_1^2 y_2,\ x_2 y_1 - 2x_1 y_2,\ x_1 x_2 + 2Ay_1 y_2 + 2y_1 y_2^2$, embed $\widetilde{M}_c$ smoothly into $\mathbb{CP}^7$ with polarization $(2, 4)$, according to (8). Then the system (15) is algebraic complete integrable and the corresponding flow evolves on an abelian surface $\widetilde{M}_c = \mathbb{C}^2/\text{lattice}$, where the lattice is generated by the period matrix $\begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}$, $\operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0$.

Theorem 2. *The abelian surface $\widetilde{M}_c$ which completes the affine surface M_c is the dual Prym variety* $\text{Prym}^*(\mathcal{D}/\mathcal{E})$ *of the genus* 3 *hyperelliptic curve $\mathcal{D}$ (16) for the involution σ interchanging the sheets of the double covering ρ (19) and the problem linearizes on this variety.*

Proof. Let $(a_1, a_2, a_3, b_1, b_2, b_3)$ be a canonical homology basis of $\mathcal{D}$ such that $\sigma(a_1) = a_3, \sigma(b_1) = b_3, \sigma(a_2) = -a_2, \sigma(b_2) = -b_2$, for the involution σ (17). As a basis of holomorphic differentials $\omega_0, \omega_1, \omega_2$ on the curve $\mathcal{D}$ (16) we take the differentials

$$\omega_1 = \frac{\alpha^2 d\alpha}{\beta}, \qquad \omega_2 = \frac{d\alpha}{\beta}, \qquad \omega_3 = \frac{\alpha d\alpha}{\beta},$$

and obviously $\sigma^*(\omega_1) = -\omega_1, \sigma^*(\omega_2) = -\omega_2, .\sigma^*(\omega_3) = \omega_3$. We consider the period matrix Ω of Jac(D)

$$\Omega = \begin{pmatrix} \int_{a_1}\omega_1 & \int_{a_2}\omega_1 & \int_{a_3}\omega_1 & \int_{b_1}\omega_1 & \int_{b_2}\omega_1 & \int_{b_3}\omega_1 \\ \int_{a_1}\omega_2 & \int_{a_2}\omega_2 & \int_{a_3}\omega_2 & \int_{b_1}\omega_2 & \int_{b_2}\omega_2 & \int_{b_3}\omega_2 \\ \int_{a_1}\omega_3 & \int_{a_2}\omega_3 & \int_{a_3}\omega_3 & \int_{b_1}\omega_3 & \int_{b_2}\omega_3 & \int_{b_3}\omega_3 \end{pmatrix}.$$

By Theorem 1,

$$\Omega = \begin{pmatrix} \int_{a_1}\omega_1 & \int_{a_2}\omega_1 & -\int_{a_1}\omega_1 & \int_{b_1}\omega_1 & \int_{b_2}\omega_1 & -\int_{b_1}\omega_1 \\ \int_{a_1}\omega_2 & \int_{a_2}\omega_2 & -\int_{a_1}\omega_2 & \int_{b_1}\omega_2 & \int_{b_2}\omega_2 & -\int_{b_1}\omega_2 \\ \int_{a_1}\omega_3 & 0 & \int_{a_1}\omega_3 & \int_{b_1}\omega_3 & 0 & \int_{b_1}\omega_3 \end{pmatrix},$$

and therefore the period matrices of Jac$(\mathcal{E})$ (i.e., $\mathcal{E}$), Prym$(\mathcal{D}/\mathcal{E})$ and Prym$^*(\mathcal{D}/\mathcal{E})$ are respectively $\Delta = (\int_{a_1}\omega_3 \;\; \int_{b_1}\omega_3)$,

$$\Gamma = \begin{pmatrix} 2\int_{a_1}\omega_1 & \int_{a_2}\omega_1 & 2\int_{b_1}\omega_1 & \int_{b_2}\omega_1 \\ 2\int_{a_1}\omega_2 & \int_{a_2}\omega_2 & 2\int_{b_1}\omega_2 & \int_{b_2}\omega_2 \end{pmatrix},$$

and

$$\Gamma^* = \begin{pmatrix} \int_{a_1}\omega_1 & \int_{a_2}\omega_1 & \int_{b_1}\omega_1 & \int_{b_2}\omega_1 \\ \int_{a_1}\omega_2 & \int_{a_2}\omega_2 & \int_{b_1}\omega_2 & \int_{b_2}\omega_2 \end{pmatrix}.$$

Let

$$L_\Omega = \left\{ \sum_{i=1}^{3} m_i \int_{a_i} \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} + n_i \int_{b_i} \begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} : m_i, n_i \in \mathbb{Z} \right\},$$

be the period lattice associated to Ω. Let us denote also by L_Δ, the period lattice associated Δ. According to Theorem 1, we get the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathcal{E} & & \mathcal{D} & & \\
 & & & & \downarrow \varphi^* & \swarrow & \downarrow \varphi & & \\
0 & \longrightarrow & \ker N_\varphi & \longrightarrow & \text{Prym}(\mathcal{D}/\mathcal{E}) \oplus \mathcal{E} = \text{Jac}(\mathcal{D}) & \xrightarrow{N_\varphi} & \mathcal{E} & \longrightarrow & 0 \\
 & & & \searrow \tau & \downarrow & & & & \\
 & & & & \widetilde{M}_c = M_c \cup 2\mathcal{D} \simeq \mathbb{C}^2/\text{lattice} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

The polarization map $\tau : \text{Prym}(\mathcal{D}/\mathcal{E}) \longrightarrow \widetilde{M}_c = \text{Prym}^*(\mathcal{D}/\mathcal{E})$, has kernel $(\varphi^*\mathcal{E}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and the induced polarization on Prym$(\mathcal{D}/\mathcal{E})$ is of type (1,2). Let

$$\widetilde{M}_c \to \mathbb{C}^2/L_\Lambda : p \curvearrowright \int_{p_0}^{p} \begin{pmatrix}\mathrm{d}t_1\\ \mathrm{d}t_2\end{pmatrix},$$

be the uniformizing map, where $\mathrm{d}t_1, \mathrm{d}t_2$ are two differentials on $\widetilde{M}_c$ corresponding to the flows generated respectively by H_1, H_2 such that: $\mathrm{d}t_1|_{\mathcal{D}} = \omega_1$ and $\mathrm{d}t_2|_{\mathcal{D}} = \omega_2$,

$$L_\Lambda = \left\{ \sum_{k=1}^{4} n_k \begin{pmatrix} \int_{\nu_k} \mathrm{d}t_1 \\ \int_{\nu_k} \mathrm{d}t_2 \end{pmatrix} : n_k \in \mathbb{Z} \right\},$$

is the lattice associated to the period matrix

$$\Lambda = \begin{pmatrix} \int_{\nu_1} dt_1 & \int_{\nu_2} dt_1 & \int_{\nu_4} dt_1 & \int_{\nu_4} dt_1 \\ \int_{\nu_1} dt_2 & \int_{\nu_2} dt_2 & \int_{\nu_3} dt_2 & \int_{\nu_4} dt_2 \end{pmatrix},$$

and $(\nu_1, \nu_2, \nu_3, \nu_4)$ is a basis of $H_1(\widetilde{M}_c, \mathbb{Z})$. By the Lefschetz theorem on hyperplane section [9], the map $H_1(\mathcal{D}, \mathbb{Z}) \longrightarrow H_1(\widetilde{M}_c, \mathbb{Z})$ induced by the inclusion $\mathcal{D} \hookrightarrow \widetilde{M}_c$ is surjective and consequently we can find 4 cycles $\nu_1, \nu_2, \nu_3, \nu_4$ on the curve $\mathcal{D}$ such that

$$\Lambda = \begin{pmatrix} \int_{\nu_1} \omega_1 & \int_{\nu_2} \omega_1 & \int_{\nu_4} \omega_1 & \int_{\nu_4} \omega_1 \\ \int_{\nu_1} \omega_2 & \int_{\nu_2} \omega_2 & \int_{\nu_3} \omega_2 & \int_{\nu_4} \omega_2 \end{pmatrix},$$

and

$$L_\Lambda = \left\{ \sum_{k=1}^{4} n_k \begin{pmatrix} \int_{\nu_k} \omega_1 \\ \int_{\nu_k} \omega_2 \end{pmatrix} : n_k \in \mathbb{Z} \right\}.$$

The cycles $\nu_1, \nu_2, \nu_3, \nu_4$ in D which we look for are a_1, b_1, a_2, b_2 and they generate $H_1(\widetilde{M}_c, \mathbb{Z})$ such that

$$\Lambda = \begin{pmatrix} \int_{a_1} \omega_1 & \int_{b_1} \omega_1 & \int_{a_2} \omega_1 & \int_{b_2} \omega_1 \\ \int_{a_1} \omega_2 & \int_{b_1} \omega_2 & \int_{a_2} \omega_2 & \int_{b_2} \omega_2 \end{pmatrix},$$

is a Riemann matrix. We show that $\Lambda = \Gamma^*$,i.e., the period matrix of $\text{Prym}^*(\mathcal{D}/\mathcal{E})$ dual of $\text{Prym}(\mathcal{D}/\mathcal{E})$. Consequently $\widetilde{M}_c$ and $\text{Prym}^*(\mathcal{D}/\mathcal{E})$ are two abelian varieties analytically isomorphic to the same complex torus $\mathbb{C}^2/L_\Lambda$. By Chow's theorem, $\widetilde{\mathcal{A}}_c$ and $\text{Prym}^*(\mathcal{D}/\mathcal{E})$ are then algebraically isomorphic. □

3.3. The Kowalewski rigid body motion

The motion for the Kowalewski's top is governed by the equations

$$\dot{m} = m \wedge \lambda m + \gamma \wedge l, \qquad \dot{\gamma} = \gamma \wedge \lambda m, \tag{20}$$

where m, γ and l denote respectively the angular momentum, the directional cosine of the z-axis (fixed in space), the center of gravity which after some rescaling and normalization may be taken as $l = (1, 0, 0)$ and $\lambda m = (m_1/2, m_2/2, m_3/2)$. The system (20) can be written

$$\begin{aligned} \dot{m}_1 &= m_2\, m_3, \qquad \dot{\gamma}_1 = 2\, m_3\gamma_2 - m_2\gamma_3, \\ \dot{m}_2 &= -m_1\, m_3 + 2\gamma_3, \qquad \dot{\gamma}_2 = m_1\gamma_3 - 2m_3\gamma_1, \\ \dot{m}_3 &= -2\gamma_2, \qquad \dot{\gamma}_3 = m_2\gamma_1 - m_1\gamma_2, \end{aligned} \tag{21}$$

with constants of motion

$$\begin{aligned} H_1 &= \frac{1}{2}\left(m_1^2 + m_2^2\right) + m_3^2 + 2\gamma_1 = c_1, \\ H_2 &= m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 = c_2, \\ H_3 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = c_3 = 1, \\ H_4 &= \left(\left(\frac{m_1 + \mathrm{i}m_2}{2}\right)^2 - (\gamma_1 + \mathrm{i}\gamma_2)\right)\left(\left(\frac{m_1 - \mathrm{i}m_2}{2}\right)^2 - (\gamma_1 - \mathrm{i}\gamma_2)\right) = c_4. \end{aligned} \tag{22}$$

The system (21) admits two distinct families of Laurent series solutions:

$$m_1(t) = \begin{cases} \dfrac{\alpha_1}{t} + \mathrm{i}\left(\alpha_1^2 - 2\right)\alpha_2 + \circ(t), \\ \dfrac{\alpha_1}{t} - \mathrm{i}\left(\alpha_1^2 - 2\right)\alpha_2 + \circ(t), \end{cases} \qquad \gamma_1(t) = \begin{cases} \dfrac{1}{2t^2} + \circ(t), \\ \dfrac{1}{2t^2} + \circ(t), \end{cases}$$

$$m_2(t) = \begin{cases} \dfrac{\mathrm{i}\alpha_1}{t} - \alpha_1^2\alpha_2 + \circ(t), \\ \dfrac{-\mathrm{i}\alpha_1}{t} - \alpha_1^2\alpha_2 + \circ(t), \end{cases} \qquad \gamma_2(t) = \begin{cases} \dfrac{\mathrm{i}}{2t^2} + \circ(t), \\ \dfrac{-\mathrm{i}}{2t^2} + \circ(t), \end{cases}$$

$$m_3(t) = \begin{cases} \frac{\mathrm{i}}{t} + \alpha_1\alpha_2 + \circ(t), \\ \frac{-\mathrm{i}}{t} + \alpha_1\alpha_2 + \circ(t), \end{cases} \qquad \gamma_3(t) = \begin{cases} \frac{\alpha_2}{t} + \circ(t), \\ \frac{\alpha_2}{t} + \circ(t), \end{cases}$$

which depend on 5 free parameters $\alpha_1, \ldots, \alpha_5$. By substituting these series in the constants of the motion H_i (22), one eliminates three parameters linearly, leading to algebraic relation between the two remaining parameters, which is nothing but the equation of the divisor D along which the m_i, γ_i blow up. Since the system (21) admits two families of Laurent solutions, then D is a set of two isomorphic curves of genus 3, $D = D_1 + D_{-1}$:

$$D_\varepsilon : P(\alpha_1, \alpha_2) = \left(\alpha_1^2 - 1\right)\left(\left(\alpha_1^2 - 1\right)\alpha_2^2 - P(\alpha_2)\right) + c_4 = 0, \tag{23}$$

where $P(\alpha_2) = c_1\alpha_2^2 - 2\varepsilon c_2\alpha_2 - 1$ and $\varepsilon = \pm 1$. Each of the curve D_ε is a $2-1$ ramified cover $(\alpha_1, \alpha_2, \beta)$ of elliptic curves D_ε^0:

$$D_\varepsilon^0 : \beta^2 = P^2(\alpha_2) - 4c_4\alpha_2^4, \tag{24}$$

ramified at the 4 points $\alpha_1 = 0$ covering the 4 roots of $P(\alpha_2) = 0$. It was shown by the author [14] that each divisor D_ε is ample and defines a polarization (1, 2), whereas the divisor D, of geometric genus 9, is very ample and defines a polarization (2, 4), accordind to (8). The affine surface $M_c = \bigcap_{i=1}^4 \{H_i = c_i\} \subset \mathbb{C}^6$, defined by putting the four invariants (22) of the Kowalewski flow (21) equal to generic constants, is the affine part of an abelian surface $\widetilde{M}_c$ with

$\widetilde{M}_c \setminus M_c = D =$ one genus 9 curve consisting of two genus 3 curves D_ε (23) intersecting in 4 points. Each D_ε is a double cover of an elliptic curve D_ε^0 (24) ramified at 4 points.

Moreover, the Hamiltonian flows generated by the vector fields X_{H_1} and X_{H_4} are straight lines on $\widetilde{M}_c$. The 8 functions 1 $, f_1 = m_1, f_2 = m_2, f_3 = m_3, f_4 = \gamma_3, f_5 = f_1^2 + f_2^2, f_6 = 4f_1f_4 - f_3f_5, f_7 = (f_2\gamma_1 - f_1\gamma_2)f_3 + 2f_4\gamma_2$, form a basis of the vector space of meromorphic functions on $\widetilde{M}_c$ with at worst a simple pole along D Moreover, the map

$$\widetilde{M}_c \simeq \mathbb{C}^2/\text{Lattice} \to \mathbb{CP}^7, \qquad (t_1, t_2) \mapsto [(1, f_1(t_1, t_2), \ldots, f_7(t_1, t_2))],$$

is an embedding of $\widetilde{M}_c$ into $\mathbb{CP}^7$. Following the method (Section 3.1), we obtain the following theorem:

Theorem 3. *The tori $\widetilde{M}_c$ can be identified as $\widetilde{M}_c = \mathrm{Prym}^*(D_\varepsilon/D_\varepsilon^0)$, i.e., dual of $\mathrm{Prym}(D_\varepsilon/D_\varepsilon^0)$ and the problem linearizes on this Prym variety.*

3.4. Kirchhoff's equations of motion of a solid in an ideal fluid

The Kirchhoff's equations of motion of a solid in an ideal fluid have the form

$$\begin{aligned} \dot{p}_1 &= p_2\frac{\partial H}{\partial l_3} - p_3\frac{\partial H}{\partial l_2}, & \dot{l}_1 &= p_2\frac{\partial H}{\partial p_3} - p_3\frac{\partial H}{\partial p_2} + l_2\frac{\partial H}{\partial l_3} - l_3\frac{\partial H}{\partial l_2}, \\ \dot{p}_2 &= p_3\frac{\partial H}{\partial l_1} - p_1\frac{\partial H}{\partial l_3}, & \dot{l}_2 &= p_3\frac{\partial H}{\partial p_1} - p_1\frac{\partial H}{\partial p_3} + l_3\frac{\partial H}{\partial l_1} - l_1\frac{\partial H}{\partial l_3}, \\ \dot{p}_3 &= p_1\frac{\partial H}{\partial l_2} - p_2\frac{\partial H}{\partial l_1}, & \dot{l}_3 &= p_1\frac{\partial H}{\partial p_2} - p_2\frac{\partial H}{\partial p_1} + l_1\frac{\partial H}{\partial l_2} - l_2\frac{\partial H}{\partial l_1}, \end{aligned} \tag{25}$$

where (p_1, p_2, p_3) is the velocity of a point fixed relatively to the solid, (l_1, l_2, l_3) the angular velocity of the body expressed with regard to a frame of reference also fixed relatively to the solid and H is the hamiltonian. These equations can be regarded as the equations of the geodesics of the right-invariant metric on the group $E(3) = SO(3) \times \mathbb{R}^3$ of motions of 3-dimensional euclidean space $\mathbb{R}^3$, generated by rotations and translations. Hence the motion has the trivial coadjoint orbit invariants $\langle p, p\rangle$ and $\langle p, l\rangle$. As it turns out, this is a special case of a more general system of equations written as

$$\dot{x} = x \wedge \frac{\partial H}{\partial x} + y \wedge \frac{\partial H}{\partial y}, \qquad \dot{y} = y \wedge \frac{\partial H}{\partial x} + x \wedge \frac{\partial H}{\partial y},$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ et $y = (y_1, y_2, y_3) \in \mathbb{R}^3$. The first set can be obtained from the second by putting $(x, y) = (l, p/\varepsilon)$ and letting $\varepsilon \to 0$. The latter set of equations is the geodesic flow on $SO(4)$ for a left invariant metric defined by the quadratic form H. In Clebsch's case, Eqs. (25) have the four invariants:

$$\begin{aligned} H_1 = H &= \frac{1}{2}\left(a_1p_1^2 + a_2p_2^2 + a_3p_3^2 + b_1l_1^2 + b_2l_2^2 + b_3l_3^2\right), \\ H_2 &= p_1^2 + p_2^2 + p_3^2, \\ H_3 &= p_1l_1 + p_2l_2 + p_3l_3, \\ H_4 &= \frac{1}{2}\left(b_1p_1^2 + b_2p_2^2 + b_3p_3^2 + \varrho\left(l_1^2 + l_2^2 + l_3^2\right)\right), \end{aligned} \tag{26}$$

with $\frac{a_2-a_3}{b_1}+\frac{a_3-a_1}{b_2}+\frac{a_1-a_2}{b_3}=0$, and the constant ϱ satisfies the conditions $\varrho=\frac{b_1(b_2-b_3)}{a_2-a_3}=\frac{b_2(b_3-b_1)}{a_3-a_1}=\frac{b_3(b_1-b_2)}{a_1-a_2}$. The system (25) can be written in the form (11) with $m=6$; to be precise

$$\dot{x}=f(x)\equiv J\frac{\partial H}{\partial x},\quad x=(p_1,p_2,p_3,l_1,l_2,l_3)^\intercal, \tag{27}$$

where

$$J=\begin{pmatrix}O & P\\ P & L\end{pmatrix},\qquad P=\begin{pmatrix}0 & -p_3 & p_2\\ p_3 & 0 & -p_1\\ -p_2 & p_1 & 0\end{pmatrix},\qquad L=\begin{pmatrix}0 & -l_3 & l_2\\ l_3 & 0 & -l_1\\ -l_2 & p_1 & 0\end{pmatrix}.$$

Consider points at infinity which are limit points of trajectories of the flow. In fact, there is a Laurent decomposition of such asymptotic solutions,

$$x(t)=t^{-1}\left(x^{(0)}+x^{(1)}t+x^{(2)}t^2+\cdots\right), \tag{28}$$

which depend on dim(phase space) $-1=5$ free parameters. Putting (28) into (27), solving inductively for the $x^{(k)}$, one finds at the 0th step a non-linear equation, $x^{(0)}+f(x^{(0)})=0$, and at the kth step, a linear system of equations,

$$(L-kI)x^{(k)}=\begin{cases}0 & \text{for } k=1\\ \text{quadratic polynomial in } x^{(1)},\ldots,x^{(k)} & \text{for } k\geq 1,\end{cases}$$

where L denotes the jacobian map of the non-linear equation above. One parameter appear at the 0th step, i.e., in the resolution of the non-linear equation and the 4 remaining ones at the kth step, $k=1,\ldots,4$. Taking into account only solutions trajectories lying on the invariant surface $M_c=\bigcap_{i=1}^4\{H_i(x)=c_i\}\subset\mathbb{C}^6$, we obtain one-parameter families which are parameterized by a curve. To be precise we search for the set of Laurent solutions which remain confined to a fixed affine invariant surface, related to specific values of c_1,c_2,c_3,c_4, i.e.,

$$\begin{aligned}D&=\bigcap_{i=1}^4\left\{t^0-\text{coefficient of } H_i(x(t))=c_i\right\},\\ &=\text{an algebraic curve defined by } \theta^2+c_1\beta^2\gamma^2+c_2\alpha^2\gamma^2+c_3\alpha^2\beta^2+c_4\alpha\beta\gamma=0,\end{aligned} \tag{29}$$

where θ is an arbitrary parameter and where $\alpha=x_4^{(0)}$, $\beta=x_5^{(0)}$, $\gamma=x_6^{(0)}$ parameterizes the elliptic curve

$$\mathcal{E}:\beta^2=d_1^2\alpha^2-1,\qquad \gamma^2=d_2^2\alpha^2+1, \tag{30}$$

with d_1,d_2 such that: $d_1^2+d_2^2+1=0$. The curve D is a 2-sheeted ramified covering of the elliptic curve $\mathcal{E}$. The branch points are defined by the 16 zeroes of $c_1\beta^2\gamma^2+c_2\alpha^2\gamma^2+c_3\alpha^2\beta^2+c_4\alpha\beta\gamma$ on $\mathcal{E}$. The curve D is unramified at infinity and by Hurwitz's formula, the genus of D is 9. Upon putting $\zeta\equiv\alpha^2$, the curve D can also be seen as a 4-sheeted unramified covering of the following curve of genus 3:

$$C:\left(\theta^2+c_1\beta^2\gamma^2+\left(c_2\gamma^2+c_3\beta^2\right)\zeta\right)^2-c_4^2\zeta\beta^2\gamma^2=0.$$

Moreover, the map $\tau:C\to C$, $(\theta,\zeta)\mapsto(-\theta,\zeta)$, is an involution on C and the quotient $C_0=C/\tau$ is an elliptic curve defined by

$$C_0:\eta^2=c_4^2\zeta\left(d_1^2d_2^2\zeta^2+\left(d_1^2-d_2^2\right)\zeta-1\right).$$

The curve C is a double ramified covering of C_0, $C\to C_0$, $(\theta,\eta,\zeta)\mapsto(\eta,\zeta)$,

$$C:\begin{cases}\theta^2=-c_1\beta^2\gamma^2-\left(c_2\gamma^2+c_3\beta^2\right)\zeta+\eta\\ \eta^2=c_4^2\zeta\left(d_1^2d_2^2\zeta^2+\left(d_1^2-d_2^2\right)\zeta-1\right).\end{cases}$$

Let $(a_1,a_2,a_3,b_1,b_2,b_3)$ be a canonical homology basis of C such that $\tau(a_1)=a_3$, $\tau(b_1)=b_3$, $\tau(a_2)=-a_2$ and $\tau(b_2)=-b_2$ for the involution τ. Using the Poincaré residue map [9], we show that

$$\omega_0=\frac{\mathrm{d}\zeta}{\eta},\qquad \omega_1=\frac{\zeta\mathrm{d}\zeta}{\theta\eta},\qquad \omega_2=\frac{\mathrm{d}\zeta}{\theta\eta},$$

form a basis of holomorphic differentials on C and $\tau^*(\omega_0)=\omega_0$, $\tau^*(\omega_k)=-\omega_k$ $(k=1,2)$. Haine [10] shows that the flow evolves on an abelian surface $\widetilde{M}_c\subseteq\mathbb{CP}^7$ of period matrix $\begin{pmatrix}2&0&a&c\\0&4&c&b\end{pmatrix}$, $\operatorname{Im}\begin{pmatrix}a&c\\c&b\end{pmatrix}>0$ and also identified $\widetilde{M}_c$ as Prym variety $\mathrm{Prym}(C/C_0)$.

Theorem 4. *The abelian surface $\widetilde{M}_c$ can be identified as* $\mathrm{Prym}(C/C_0)$. *More precisely*

$$\bigcap_{i=1}^4\left\{x\in\mathbb{C}^6,H_i(x)=c_i\right\}=\mathrm{Prym}(C/C_0)\setminus D,$$

where D is a genus 9 *curve* (29)*, which is a ramified cover of an elliptic curve $\mathcal{E}$* (30) *with* 16 *branch points.*

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