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# Integrable Systems and Complex Geometry

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**Abstract**—In this paper, we discuss an interaction between complex geometry and integrable systems. Section 1 reviews the classical results on integrable systems. New examples of integrable systems, which have been discovered, are based on the Lax representation of the equations of motion. These systems can be realized as straight line motions on a Jacobi variety of a so-called spectral curve. In Section 2, we study a Lie algebra theoretical method leading to integrable systems and we apply the method to several problems. In Section 3, we discuss the concept of the algebraic complete integrability (a.c.i.) of hamiltonian systems. Algebraic integrability means that the system is completely integrable in the sens of the phase space being folited by tori, which in addition are real parts of a complex algebraic tori (abelian varieties). The method is devoted to illustrate how to decide about the a.c.i. of hamiltonian systems and is applied to some examples. Finally, in Section 4 we study an a.c.i. in the generalized sense which appears as covering of a.c.i. system. The manifold invariant by the complex flow is covering of abelian variety.

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## CONTENTS



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## 1. INTEGRABLE SYSTEMS

Let $M$ be an even-dimensional differentiable manifold. A symplectic structure (or symplectic form) on $M$ is a closed non-degenerate differential 2-form $\omega$ defined everywhere on $M$. The non-degeneracy condition means that

$$\forall x \in M, \forall \xi \neq 0, \exists \eta : \omega(\xi, \eta) \neq 0, (\xi, \eta \in T_x M).$$

The pair $(M, \omega)$ is called a symplectic manifold.

**Example 1.1.** *The cotangent bundle $T^*M$ possesses in a natural way a symplectic structure. In a local coordinate $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, $2n = \dim M$, the form $\omega$ is given by $\omega = \sum_{k=1}^{n} dx_k \wedge dy_k$.*

**Example 1.2.** *Another important class of symplectic manifolds consists of the coadjoints orbits $\mathcal{O} \subset \mathcal{G}^*$, where $\mathcal{G}$ is the algebra of a Lie group $\mathcal{G}$ and $\mathcal{G}_\mu = \{Ad_g^* \mu : g \in \mathcal{G}\}$ is the orbit of $\mu \in \mathcal{G}^*$ under the coadjoint representation.*

**Theorem 1.** (a) *Let $I$: $T_x^*M \longrightarrow T_xM$, $\omega_\xi^1 \longmapsto \xi$, be a map defined by $\omega_\xi^1(\eta) = \omega(\eta, \xi)$, $\forall \eta \in T_xM$. Then $I$ is an isomorphism generated by the symplectic form $\omega$.*

(b) *The symplectic form $\omega$ induces a hamiltonian vector field $IdH$: $M \longrightarrow T_xM$, $x \longmapsto IdH(x)$, where $H$: $M \longrightarrow \mathbb{R}$, is a differentiable function (called hamiltonian). In others words, the differential system defined by*

$$\dot{x}(t) = X_H(x(t)) = IdH(x),$$

*is a hamiltonian vector field associated to the function $H$. The matrix that is associated to an hamiltonian system determine a symplectic structure.*

**Proof.** (a) Denote by $I^{-1}$ the map $I^{-1}$: $T_xM \longrightarrow T_x^*M$, $\xi \longmapsto I^{-1}(\xi) \equiv \omega_\xi^1$, with $I^{-1}(\xi)(\eta) = \omega_\xi^1(\eta) = \omega(\eta, \xi)$, $\forall \eta \in T_xM$. The fact that the form $\omega$ is bilinear implies that

$$\begin{aligned} I^{-1}(\xi_1 + \xi_2)(\eta) &= \omega(\eta, \xi_1 + \xi_2), \\ &= \omega(\eta, \xi_1) + \omega(\eta, \xi_2), \\ &= I^{-1}(\xi_1)(\eta) + I^{-1}(\xi_2)(\eta), \quad \forall \eta \in T_xM. \end{aligned}$$

Now, since $\dim T_xM = \dim T_x^*M$, to show that $I^{-1}$ is bijective, it suffices to show that is injective. The form $\omega$ is non-degenerate, it follows that

$$KerI^{-1} = \{\xi \in T_xM : \omega(\eta, \xi) = 0, \quad \forall \eta \in T_xM\} = \{0\}.$$

Hence $I^{-1}$ is an isomorphism and consequently $I$ is also an isomorphism (the inverse of an isomorphism is an isomorphism).

(b) Let $(x_1, \ldots, x_m)$ be a local coordinate system on $M$, $(m = \dim M)$. We have

$$\dot{x}(t) = \sum_{k=1}^{n} \frac{\partial H}{\partial x_k} I(dx_k) = \sum_{k=1}^{n} \frac{\partial H}{\partial x_k} \xi^k, \tag{1}$$

where $I(dx_k) = \xi^k \in T_xM$ is defined such that: $\forall \eta \in T_xM$, $\eta_k = dx_k(\eta) = \omega(\eta, \xi^k)$, ($k$th component of $\eta$). Define $(\eta_1, \ldots, \eta_m)$ and $(\xi_1^k, \ldots, \xi_m^k)$ to be respectively the components of $\eta$ and $\xi^k$, then

$$\begin{aligned}\eta_k &= \omega\left(\sum_{i=1}^m \eta_i \frac{\partial}{\partial x_i}, \sum_{j=1}^m \xi_j^k \frac{\partial}{\partial x_j}\right)\\ &= \sum_{i=1}^m \eta_i \left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\right)\xi_j^k,\\ &= (\eta_1, \ldots, \eta_m)\, J^{-1}\begin{pmatrix}\xi_1^k\\ \vdots\\ \xi_m^k\end{pmatrix},\end{aligned}$$

where $J^{-1}$ is the matrix defined by $J^{-1} \equiv \left(\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right)_{1\leq i,j\leq m}$. Since this matrix is invertible,[1] we can search $\xi^k$ such that:

$$J^{-1}\begin{pmatrix}\xi_1^k\\ \vdots\\ \xi_m^k\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\\ 1 \leftrightsquigarrow k\text{-th place}\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

The matrix $J^{-1}$ is invertible, which implies

$$\begin{pmatrix}\xi_1^k\\ \vdots\\ \xi_m^k\end{pmatrix} = J\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix},$$

from which $\xi^k = $ ($k$-th column of J), i.e., $\xi_i^k = J_{ik}$, $1 \leq i \leq m$, and consequently $\xi^k = \sum_{i=1}^m J_{ik}\dfrac{\partial}{\partial x_i}$.

[1] Indeed, it suffices to show that the matrix $J^{-1}$ has maximal rank. Suppose this were not possible, i.e., we assume that $rank(J^{-1}) \neq m$. Hence $\sum_{i=1}^m a_i\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = 0$, $\forall 1 \leq j \leq m$, with $a_i$ not all null and $\omega\left(\sum_{i=1}^m a_i\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = 0$, $\forall 1 \leq j \leq m$. In fact, since $\omega$ is non-degenerate, we have $\sum_{i=1}^m a_i\frac{\partial}{\partial x_i} = 0$. Now $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right)$ is a basis of $T_xM$, then $a_i = 0$, $\forall i$, contradiction.

It is easily verified that the matrix $J$ is skew-symmetric.[2] From (1) we deduce that

$$\dot{x}(t)=\sum_{k=1}^{m}\frac{\partial H}{\partial x_k}\sum_{i=1}^{m}J_{ik}\frac{\partial}{\partial x_i}=\sum_{i=1}^{m}\left(\sum_{k=1}^{m}J_{ik}\frac{\partial H}{\partial x_k}\right)\frac{\partial}{\partial x_i}.$$

Writing $\dot{x}(t)=\sum_{i=1}^{m}\frac{dx_i\,(t)}{dt}\frac{\partial}{\partial x_i}$, it is seen that $\dot{x}_i\,(t)=\sum_{k=1}^{m}J_{ik}\frac{\partial H}{\partial x_k}, 1\leq i\leq j\leq m$, which can be written in more compact form $\dot{x}(t)=J\,(x)\,\frac{\partial H}{\partial x}$, this is the hamiltonian vector field associated to the function $H$. This concludes the proof of the theorem.

We define a Poisson bracket (or Poisson structure) on the space $\mathcal{C}^\infty$ as

$$\{,\}:\quad \mathcal{C}^\infty\,(M)\times\mathcal{C}^\infty\,(M)\longrightarrow\mathcal{C}^\infty\,(M)\,,\quad (F,G)\longmapsto\{F,G\}\,,$$

where $\{F,G\}=d_uF\,(X_G)=X_GF\,(u)=\omega\,(X_G,X_F)$. This bracket is skew-symmetric $\{F,G\}=-\,\{G,F\}$, obeys the Leibniz rule $\{FG,H\}=F\,\{G,H\}+G\,\{F,H\}$, and satisfies the Jacobi identity

$$\{\{H,F\}\,,G\}+\{\{F,G\}\,,H\}+\{\{G,H\}\,,F\}=0.$$

When this Poisson structure is non-degenerate, we obtain the symplectic structure discussed above.

Consider now $M=\mathbb{R}^n\times\mathbb{R}^n$ and let $p\in M$. By Darboux's theorem [3], there exists a local coordinate system$(x_1,\ldots,x_n,y_1,\ldots,y_n)$ in a neighbourhood of $p$ such that

$$\{H,F\}=\sum_{i=1}^{n}\left(\frac{\partial H}{\partial x_i}\frac{\partial F}{\partial y_i}-\frac{\partial H}{\partial y_i}\frac{\partial F}{\partial x_i}\right).$$

Then $X_H=\sum_{i=1}^{n}\left(\frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i}-\frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i}\right)$, and $X_HF=\{H,F\}, \forall F\in\mathcal{C}^\infty\,(M)$. A nonconstant function $F$ is called an integral (first integral or constant of motion) of $X_F$, if $X_HF=0$. In particular, $H$ is integral. Two functions $F$ and $G$ are said to be in involution or to commute, if $\{F,G\}=0$. The hamiltonian systems form a Lie algebra.

We now give the following definition of the Poisson bracket:

$$\{F,G\}=\left\langle\frac{\partial F}{\partial x},J\frac{\partial G}{\partial x}\right\rangle=\sum_{i,j}J_{ij}\frac{\partial F}{\partial x_i}\frac{\partial G}{\partial x_j}.$$

After some algebraic manipulation, we deduce that If

$$\sum_{k=1}^{2n}\left(J_{kj}\frac{\partial J_{li}}{\partial x_k}+J_{ki}\frac{\partial J_{jl}}{\partial x_k}+J_{kl}\frac{\partial J_{ij}}{\partial x_k}\right)=0,\quad \forall 1\leq i,j,l\leq 2n,$$

then $J$ satisfies the Jacobi identity.

Consequently, we have a complete characterization of hamiltonian vector field

$$\dot{x}(t)=X_H\,(x\,(t))=J\frac{\partial H}{\partial x},\quad x\in M, \tag{2}$$

where $H$: $M\longrightarrow\mathbb{R}$, is a differentiable function (the hamiltonian) and $J=J\,(x)$ is a skew-symmetric matrix, possibly depending on $x\in M$, for which the corresponding Poisson bracket satisfies the Jacobi identity:

$$\{\{H,F\}\,,G\}+\{\{F,G\}\,,H\}+\{\{G,H\}\,,F\}=0,$$

with $\{H,F\}=\left\langle\frac{\partial H}{\partial x},J\frac{\partial F}{\partial x}\right\rangle=\sum_{i,j}J_{ij}\frac{\partial H}{\partial x_i}\frac{\partial F}{\partial x_j}$, the Poisson bracket.

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[2] Indeed, since $\omega$ is symmetric i.e., $\omega\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=-\omega\left(\frac{\partial}{\partial x_j},\frac{\partial}{\partial x_i}\right)$, it follows that $J^{-1}$ is skew-symmetric. Then, $I=J.J^{-1}=\left(J^{-1}\right)^\top.J^\top=-J^{-1}.J$, and consequently $J^\top=J$.

**Example 1.3.** *An important special case is when* $J = \begin{pmatrix} O & -I \\ I & O \end{pmatrix}$, *where* $I$ *is the* $n \times n$ *identity matrix. The condition on* $J$ *is trivially satisfied. Indeed, here the matrix* $J$ *do not depend on the variable* $x$ *and we have*

$$\{H, F\} = \sum_{i=1}^{2n} \frac{\partial H}{\partial x_i} \sum_{j=1}^{2n} J_{ij} \frac{\partial F}{\partial x_j} = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial x_{n+i}} \frac{\partial F}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial F}{\partial x_{n+i}} \right).$$

*Moreover, equations (10) are transformed into*

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \ldots, \dot{q}_n = \frac{\partial H}{\partial p_n}, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \ldots, \dot{p}_n = -\frac{\partial H}{\partial q_n},$$

or $q_1 = x_1, \ldots, q_n = x_n, p_1 = x_{n+1}, \ldots, p_n = x_{2n}$. *These are exactly the well known differential equations of classical mechanics in canonical form.*

It is a fundamental and important problem to investigate the integrability of hamiltonian systems. Recently there has been much effort given for finding integrable hamiltonian systems, not only because they have been on the subject of powerful and beautiful theories of mathematics, but also because the concepts of integrability have been applied to an increasing number of applied sciences. The so-called Arnold-Liouville theorem play a crucial role in the study of such systems; the regular compact level manifolds defined by the intersection of the constants of motion are diffeomorphic to a real torus on which the motion is quasi-periodic as a consequence of the following purely differential geometric fact: a compact and connected $n$-dimensional manifold on which there exist $n$ vector fields which commute and are independent at every point is diffeomorphic to an $n$-dimensional real torus and each vector field will define a linear flow there.

**Theorem 2.** *(Arnold-Liouville theorem)* [3, 15]: *Let* $H_1 = H, H_2, \ldots, H_n$, *be* $n$ *first integrals on a 2n-dimensional symplectic manifold that are functionally independent (i.e.,* $dH_1 \wedge \cdots \wedge dH_n \neq 0$*), and pairwise in involution. For generic* $c = (c_1, \ldots, c_n)$ *the level set*

$$M_c = \bigcap_{i=1}^{n} \{x \in M : H_i(x) = c_i, \quad c_i \in \mathbb{R}\},$$

*will be an n-manifold. If* $M_c$ *is compact and connected, it is diffeomorphic to an* $n$*-dimensional torus* $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ *and the solutions of the system (2) are then straight-line motions on* $\mathbb{T}^n$. *If* $M_c$ *is not compact but the flow of each of the vector fields* $X_{H_k}$ *is complete on* $M_c$, *then* $M_c$ *is diffeomorphic to a cylinder* $\mathbb{R}^k \times \mathbb{T}^{n-k}$ *under which the vector fields* $X_{H_k}$ *are mapped to linear vector fields.*

As a consequence, we obtain the concept of complete integrability of a hamiltonian system. For the sake of clarity, we shall distinguish two cases:

(a) **Case** 1: *dét* $J \neq 0$. The rank of the matrix $J$ is even, $m = 2n$. A hamiltonian system (2) is completely integrable or Liouville-integrable if there exist $n$ firsts integrals $H_1 = H, H_2, \ldots, H_n$ in involution, i.e., $\{H_k, H_l\} = 0, 1 \leq k, l \leq n$, with linearly independent gradients, i.e., $dH_1 \wedge \cdots \wedge dH_n \neq 0$. For generic $c = (c_1, \ldots, c_n)$ the level set

$$M_c = \bigcap_{i=1}^{n} \{x \in M : H_i(x) = c_i, \quad c_i \in \mathbb{R}\},$$

will be an $n$-manifold. By the Arnold-Liouville theorem, if $M_c$ is compact and connected, it is diffeomorphic to an $n$-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and each vector field will define a linear flow there. In some open neighbourhood of the torus there are coordinates $s_1, \ldots, s_n, \varphi_1, \ldots, \varphi_n$ in which $\omega$ takes the form $\omega = \sum_{k=1}^{n} ds_k \wedge d\varphi_k$. Here the functions $s_k$ (called action-variables) give coordinates in the direction transverse to the torus and can be expressed functionally in terms of the firsts integrals $H_k$. The functions $\varphi_k$ (called angle-variables) give standard angular coordinates on the torus, and every vector field $X_{H_k}$

can be written in the form $\dot{\varphi}_k = h_k(s_1, \ldots, s_n)$, that is, its integral trajectories define a conditionally-periodic motion on the torus. In a neighbourhood of the torus the hamiltonian vector field $X_{H_k}$ take the following form $\dot{s}_k = 0, \dot{\varphi}_k = h_k(s_1, \ldots, s_n)$, and can be solved by quadratures.

(b) **Case** 2: *dét* $J = 0$. We reduce the problem to $m = 2n + k$ and we look for $k$ Casimir functions (or trivial invariants) $H_{n+1}, \ldots, H_{n+k}$, leading to identically zero hamiltonian vector fields $J\dfrac{\partial H_{n+i}}{\partial x} = 0$, $1 \leq i \leq k$. In other words, the system is hamiltonian on a generic symplectic manifold

$$\bigcap_{i=n+1}^{n+k} \{x \in \mathbb{R}^m : H_i(x) = c_i\},$$

of dimension $m - k = 2n$. If for most values of $c_i \in \mathbb{R}$, the invariant manifolds

$$\bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m : H_i(x) = c_i\},$$

are compact and connected, then they are n-dimensional tori $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ by the Arnold-Liouville theorem and the hamiltonian flow is linear in angular coordinates of the torus.

## 2. ISOSPECTRAL DEFORMATION METHOD

A Lax equation is given by a differential equation of the form

$$\dot{A}(t) = [A(t), B(t)] \text{ or } [B(t), A(t)], \tag{3}$$

where

$$A(t) = \sum_{k=1}^{N} A_k(t) h^k, \quad B(t) = \sum_{k=1}^{N} B_k(t) h^k,$$

are functions depending on a parameter $h$ (spectral parameter) whose coefficients $A_k$ and $B_k$ are matrices in Lie algebras. The pair $(A, B)$ is called Lax pair. This equation established a link between the Lie group theoretical and the algebraic geometric approaches to complete integrability. The solution to (3) has the form $A(t) = g(t)A(0)g(t)^{-1}$, where $g(t)$ is a matrix defined as $\dot{g}(t) = -A(t)g(t)$. We form the polynomial $P(h, z) = \det(A - zI)$, where $z$ is another variable and $I$ the $n \times n$ identity matrix. We define the curve (spectral curve) $\mathcal{C}$, to be the normalization of the complete algebraic curve whose affine equation is $P(h, z) = 0$.

**Theorem 3.** *The polynomial $P(h, z)$ is independent of $t$. Moreover, the functions $tr(A^n)$ are first integrals for (3).*

**Proof.** Let us call $L \equiv A - zI$. Observe that

$$\dot{P} = \det L.tr\left(L^{-1}\dot{L}\right) = \det L.tr\left(L^{-1}BL - B\right) = 0,$$

since $trL^{-1}BL = trB$. On the other hand

$$\begin{aligned}
\dot{A}^n &= \dot{A}A^{n-1} + A\dot{A}A^{n-2} + \cdots + A^{n-1}\dot{A} \\
&= [A, B] A^{n-1} + A[A, B] A^{n-2} + \cdots + A^{n-1}[A, B] \\
&= (AB - BA) A^{n-1} + \cdots + A^{n-1}(AB - BA) \\
&= ABA^{n-1} - BA^n + \cdots + A^nB - A^{n-1}BA \\
&= A\left(BA^{n-1}\right) - \left(BA^{n-1}\right)A + \cdots + A\left(A^{n-1}B\right) - \left(A^{n-1}B\right)A.
\end{aligned}$$

Since $tr(X + Y) = trX + trY$, $trXY = trYX$, $X, Y \in \mathcal{M}_n(\mathbb{C})$, we obtain

$$\frac{d}{dt}tr(A_h^n) = tr\frac{d}{dt}(A_h^n) = 0,$$

and consequently $tr\,(A^n)$ are first integrals of motion. This ends the proof of the theorem.

We have shown that a hamiltonian flow of the type (3) preserves the spectrum of $A$ and therefore its characteristic polynomial. The curve $\mathcal{C}$: $P(z,h)=\det(A(h)-zI)=0$, is time independent, i.e., its coefficients $tr\,(A^n)$ are integrals of the motion (equivalently, $A(t)$ undergoes an isospectral deformation. Some hamiltonian flows on Kostant-Kirillov coadjoint orbits in subalgebras of infinite dimensional Lie algebras (Kac-Moody Lie algebras) yield large classes of extended Lax pairs (3). A general statement leading to such situations is given by the Adler-Kostant-Symes theorem.

**Theorem 4.** *Let $\mathcal{L}$ be a Lie algebra paired with itself via a nondegenerate, ad-invariant bilinear form $\langle\ ,\ \rangle$, $\mathcal{L}$ having a vector space decomposition $\mathcal{L}=\mathcal{K}+\mathcal{N}$ with $\mathcal{K}$ and $\mathcal{N}$ Lie subalgebras. Then, with respect to $\langle\ ,\ \rangle$, we have the splitting $\mathcal{L}=\mathcal{L}^*=\mathcal{K}^\perp+\mathcal{N}^\perp$ and $\mathcal{N}^*=\mathcal{K}^\perp$ paired with $\mathcal{N}$ via an induced form $\langle\langle,\rangle\rangle$ inherits the coadjoint symplectic structure of Kostant and Kirillov; its Poisson bracket between functions $H_1$ and $H_2$ on $\mathcal{N}^*$ reads*

$$\{H_1,H_2\}(a)=\langle\langle a,[\nabla_{\mathcal{N}^*}H_1,\nabla_{\mathcal{N}^*}H_2]\rangle\rangle,\quad a\in\mathcal{N}^*.$$

*Let $V\subset\mathcal{N}^*$ be an invariant manifold under the above co-adjoint action of $\mathcal{N}$ on $\mathcal{N}^*$ and let $\mathcal{A}(V)$ be the algebra of functions defined on a neighborhood of $V$, invariant under the coadjoint action of $\mathcal{L}$ (which is distinct from the $\mathcal{N}-\mathcal{N}^*$ action). Then the functions $H$ in $\mathcal{A}(V)$ lead to commuting Hamiltonian vector fields of the Lax isospectral form*

$$\dot a=[a,pr_{\mathcal{K}}(\nabla H)],\quad pr_{\mathcal{K}}\quad projection\quad onto\quad \mathcal{K}.$$

This theorem produces hamiltonian systems having many commuting integrals ; some precise results are known for interesting classes of orbits in both the case of finite and infinite dimensional Lie algebras. Any finite dimensional Lie algebra $\mathcal{L}$ with bracket $[,]$ and killing form $\langle,\rangle$ leads to an infinite dimensional formal Laurent series extension $\mathcal{L}=\sum_{-\infty}^{N}A_ih^i : A_i\in\mathcal{L}$, $N\in\mathbb{Z}$ free, with bracket $\left[\sum A_ih^i,\sum B_jh^j\right]=\sum_{i,j}[A_i,B_j]\,h^{i+j}$, and ad-invariant, symmetric forms $\left\langle\sum A_ih^i,\sum B_jh^j\right\rangle_k=\sum_{i+j=-k}\langle A_i,B_j\rangle$, depending on $k\in\mathbb{Z}$. The forms $\langle,\rangle_k$ are non degenerate if $\langle,\rangle$ is so. Let $\mathcal{L}_{p,q}$ $(p\le q)$ be the vector space of powers of $h$ between $p$ and $q$. A first interesting class of problems is obtained by taking $\mathcal{L}=\mathcal{G}l(n,\mathbb{R})$ and by putting the form $\langle,\rangle_1$ on the Kac-Moody extension. Then we have the decomposition into Lie subalgebras $\mathcal{L}=\mathcal{L}_{0,\infty}+\mathcal{L}_{-\infty,-1}=\mathcal{K}+\mathcal{N}$ with $\mathcal{K}=\mathcal{K}^\perp$, $\mathcal{N}=\mathcal{N}^\perp$ and $\mathcal{K}=\mathcal{N}^*$. Consider the invariant manifold $V_m$, $m\ge1$ in $\mathcal{K}=\mathcal{N}^*$, defined as

$$V_m=\left\{A=\sum_{i=1}^{m-1}A_ih^i+\alpha h^m,\quad \alpha=\mathrm{diag}(\alpha_1,\cdots,\alpha_n)\text{ fixed}\right\},$$

with $\mathrm{diag}\,(A_{m-1})=0$.

**Theorem 5.** *The manifold $V_m$ has a natural symplectic structure, the functions $H=\left\langle f(Ah^{-j}),h^k\right\rangle_1$ on $V_m$ for good functions $f$ lead to complete integrable commuting hamiltonian systems of the form*

$$\dot A=\left[A,pr_{\mathcal{K}}(f'(Ah^{-j})h^{k-j})\right],\quad A=\sum_{i=0}^{m-1}A_ih^i+\alpha h,$$

*and their trajectories are straight line motions on the jacobian of the curve $\mathcal{C}$ of genus $(n-1)(nm-2)/2$ defined by $P(z,h)=\det(A-zI)=0$. The coefficients of this polynomial provide the orbit invariants of $V_m$ and an independent set of integrals of the motion (of particular interest are the flows where $j=m,k=m+1$ which have the following form*

$$\dot A=\left[A,ad_\beta ad_\alpha^{-1}A_{m-1}+\beta h\right],\quad \beta_i=f'(\alpha_i),$$

*the flow depends on $f$ through the relation $\beta_i=f'(\alpha_i)$ only).*

Another class is obtained by choosing any semi-simple Lie algebra $L$. Then the Kac-Moody extension $\mathcal{L}$ equipped with the form $\langle,\rangle=\langle,\rangle_0$ has the natural level decomposition $\mathcal{L}=\sum_{i\in\mathbb{Z}}L_i, [L_i,L_j]\subset$

$L_{i+j}$, $[L_0, L_0] = 0$, $L_i^* = L_{-i}$. Let $B^+ = \sum_{i\geq 0} L_i$ and $B^- = \sum_{i\langle 0} L_i$. Then the product Lie algebra $\mathcal{L} \times \mathcal{L}$ has the following bracket and pairing

$$\left[(l_1, l_2), (l_1', l_2')\right] = \left([l_1, l_1'], -[l_2, l_2']\right), \quad \left\langle (l_1, l_2), (l_1', l_2')\right\rangle = \langle l_1, l_1'\rangle - \langle l_2, l_2'\rangle.$$

It admits the decomposition into $\mathcal{K} + \mathcal{N}$ with

$$\mathcal{K} = \{(l, -l) : l \in \mathcal{L}\}, \quad \mathcal{K}^\perp = \{(l, l) : l \in \mathcal{L}\},$$

$$\mathcal{N} = \left\{(l_-, l_+) : l_- \in B^-, l_+ \in B^+, pr_0(l_-) = pr_0(l_+)\right\},$$

$$\mathcal{N}^\perp = \left\{(l_-, l_+) : l_- \in B^-, l_+ \in B^+, pr_0(l_+ + l_-) = 0\right\},$$

where $pr_0$ denotes projection onto $L_0$. Then from the last theorem, the orbits in $\mathcal{N}^*$=$K^\perp$ possesses a lot of commuting hamiltonian vector fields of Lax form:

**Theorem 6.** *The N-invariant manifolds $V_{-j,k} = \sum_{-j\leq i\leq k} L_i \subseteq \mathcal{L} \simeq \mathcal{K}^\perp$, has a natural symplectic structure and the functions $H(l_1, l_2) = f(l_1)$ on $V_{-j,k}$ lead to commuting vector fields of the Lax form*

$$\dot{l} = \left[l, \left(pr^+ - \frac{1}{2}pr_0\right)\nabla H\right], \quad pr^+ \quad projection \quad onto\ B^+,$$

*their trajectories are straight line motions on the Jacobian of a curve defined by the characteristic polynomial of elements in $V_{-j,k}$.*

Using the van Moerbeke-Mumford linearization method [21], Adler and van Moerbeke [1] showed that the linearized flow could be realized on the jacobian variety $Jac(\mathcal{C})$ (or some sub-abelian variety of it) of the algebraic curve (spectral curve) $\mathcal{C}$ associated to (3). We then construct an algebraic map from the complex invariant manifolds of these hamiltonian systems to the jacobian variety $Jac(\mathcal{C})$ of the curve $\mathcal{C}$. Therefore all the complex flows generated by the constants of the motion are straight line motions on these jacobian varieties i.e. the linearizing equations are given by

$$\int_{s_1(0)}^{s_1(t)} \omega_k + \int_{s_2(0)}^{s_2(t)} \omega_k + \ldots + \int_{s_g(0)}^{s_g(t)} \omega_k = c_k t, \quad 0 \leq k \leq g,$$

where $\omega_1, \ldots, \omega_g$ span the $g$-dimensional space of holomorphic differentials on the curve $\mathcal{C}$ of genus $g$. In an unifying approach, Griffiths [8] has found necessary and sufficient conditions on $B$ for the Lax flow (3) to be linearizable on the jacobi variety of its spectral curve, without reference to Kac-Moody Lie algebras.

Next I schall discuss a number of integrable hamiltonian systems.

### *2.1. The Euler Rigid Body Motion*

It express the free motion of a rigid body around a fixed point. Let $M = (m_1, m_2, m_3)$ be the angular momentum, $\Omega = (m_1/I_1, m_2/I_2, m_3/I_3)$ the angular velocity and $I_1, I_2$ et $I_3$, the principal moments of inertia about the principal axes of inertia. Then the motion of the body is governed by

$$\dot{M} = M \wedge \Omega. \tag{4}$$

If one identifies vectors in $\mathbb{R}^3$ with skew-symmetric matrices by the rule

$$a = (a_1, a_2, a_3), \quad A = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix},$$

then $a \wedge b \longmapsto [A,B] = AB - BA$. Using this isomorphism between $(\mathbb{R}^3, \wedge)$ and $(so(3), [,])$, we write (4) as $\dot{M} = [M, \Omega]$, where

$$M = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \in so(3), \quad \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \in so(3).$$

Now $M = I\Omega$, this implies that

$$\dot{M} = [M, \Lambda M], \tag{5}$$

where

$$\Lambda M = \begin{pmatrix} 0 & -\lambda_3 m_3 & \lambda_2 m_2 \\ \lambda_3 m_3 & 0 & -\lambda_1 m_1 \\ -\lambda_2 m_2 & \lambda_1 m_1 & 0 \end{pmatrix} \in so(3),$$

with $\lambda_i \equiv I_i^{-1}$. Equation (5) is explicitly given by

$$\dot{m}_1 = (\lambda_3 - \lambda_2)\, m_2 m_3, \quad \dot{m}_2 = (\lambda_1 - \lambda_3)\, m_1 m_3, \quad \dot{m}_3 = (\lambda_2 - \lambda_1)\, m_1 m_2, \tag{6}$$

and can be written as a hamiltonian vector field

$$\dot{x} = J\frac{\partial H}{\partial x}, \quad x = (m_1, m_2, m_3)^\intercal,$$

with the hamiltonian $H = \frac{1}{2}\left(\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2\right)$, and

$$J = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \in so(3).$$

We have $\det J = 0$, so $m = 2n + k$ and $m - k = rk\ J$. Here $m = 3$ and $rk\ J = 2$, then $n = k = 1$. The system (6) has beside the energy $H_1 = H$, a trivial invariant $H_2$, i.e., such that: $J\frac{\partial H_2}{\partial x} = 0$, or

$$\begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H_2}{\partial m_1} \\ \frac{\partial H_2}{\partial m_2} \\ \frac{\partial H_2}{\partial m_3} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

implying $\frac{\partial H_2}{\partial m_1} = m_1$, $\frac{\partial H_2}{\partial m_2} = m_2$, $\frac{\partial H_2}{\partial m_3} = m_3$, and consequently

$$H_2 = \frac{1}{2}\left(m_1^2 + m_2^2 + m_3^2\right).$$

The system evolves on the intersection of the sphere $H_1 = c_1$ and the ellipsoid $H_2 = c_2$. In $\mathbb{R}^3$, this intersection will be isomorphic to two circles $\left(\text{with } \frac{c_2}{\lambda_3} < c_1 < \frac{c_2}{\lambda_1}\right)$. We shall show that the problem can be integrated in terms of elliptic functions, as Euler discovered using his then newly invented theory of elliptic integrals. Observe that the first equation of (6) reads

$$\frac{dm_1}{m_2 m_3} = (\lambda_3 - \lambda_2)\, dt, \tag{7}$$

where $m_1, m_2$ and $m_3$ are related by

$$\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2 = c_1, \quad m_1^2 + m_2^2 + m_3^2 = c_2.$$

Therefore, if $\lambda_2 \neq \lambda_3$, we have

$$m_2 = \pm\sqrt{\frac{c_2\lambda_3 - c_1 + (\lambda_1 - \lambda_3)\, m_1^2}{\lambda_3 - \lambda_2}}, \quad m_3 = \pm\sqrt{\frac{c_1 - c_2\lambda_2 + (\lambda_2 - \lambda_1)\, m_1^2}{\lambda_3 - \lambda_2}}.$$

Substituting these expressions into (7), we find after integration that the system (6) amounts to an elliptic integral

$$\int\limits_{m_1(0)}^{m_1(t)} \frac{dm}{\sqrt{(m^2 + a)\,(m^2 + b)}} = ct,$$

with respect to the elliptic curve

$$\mathcal{C}: \quad w^2 = \left(z^2 + a\right)\left(z^2 + b\right), \tag{8}$$

with $a = \dfrac{c_2\lambda_3 - c_1}{\lambda_1 - \lambda_3}$, $b = \dfrac{c_1 - c_2\lambda_2}{\lambda_2 - \lambda_1}$, $c = \sqrt{(\lambda_1 - \lambda_3)\,(\lambda_2 - \lambda_1)}$. Then the functions $m_i(t)$ can be expressed in terms of theta-functions of $t$, according to the classical inversion of abelian integrals.

We shall use the Lax representation of the equations of motion to show that the linearized Euler flow can be realized on an elliptic curve isomorphic to the original elliptic curve (8). The solution to (5) has the form

$$M\,(t) = O\,(t)\, M\,(t)\, M^\top\,(t)\,,$$

where $O\,(t)$ is one parameter sub-group of $SO\,(3)$. So the hamiltonian flow (5) preserves the spectrum of $X$ and therefore its characteristic polynomial $\det\,(M - zI) = -z\left(z^2 + m_1^2 + m_2^2 + m_3^2\right)$. Unfortunately, the spectrum of a $3 \times 3$ skew- symmetric matrix provides only one piece of information; the conservation of energy does not appear as part of the spectral information. Therefore one is let to considering another formulation. The basic observation, due to Manakov [20], is that equation (5) is equivalent to the Lax equation

$$\dot{A} = [A, B]\,,$$

where $A = M + \alpha h$, $B = \Lambda M + \beta h$, with a formal indeterminate $h$ and

$$\alpha = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}, \quad \beta = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix},$$

$$\lambda_1 = \frac{\beta_3 - \beta_2}{\alpha_3 - \alpha_2}, \quad \lambda_2 = \frac{\beta_1 - \beta_3}{\alpha_1 - \alpha_3}, \quad \lambda_3 = \frac{\beta_2 - \beta_1}{\alpha_2 - \alpha_1},$$

and all $\alpha_i$ distinct. The characteristic polynomial of $A$ is

$$\begin{aligned} P\,(h, z) &= \det\,(A - zI) \\ &= \det\,(M + \alpha h - zI) \\ &= \prod_{j=1}^{3} (\alpha_j h - z) + \left(\sum_{j=1}^{3} \alpha_j m_j^2\right) h - \left(\sum_{j=1}^{3} m_j^2\right) z. \end{aligned}$$

The spectrum of the matrix $A = M + \alpha h$ as a function of $h \in \mathbb{C}$ is time independent and is given by the zeroes of the polynomial $P(h, z)$, thus defining an algebraic curve (spectral curve). Letting $w = h/z$, we obtain the following elliptic curve

$$z^2 \prod_{j=1}^{3} (\alpha_j w - 1) + 2H_1 w - 2H_2 = 0,$$

which is shown to be isomorphic to the original elliptic curve. Finally, we have the

**Theorem 7.** *The Euler rigid body motion is a completely integrable system and the linearized flow can be realized on an elliptic curve.*

### 2.2. The Geodesic Flow for a Left Invariant Metric on $SO(4)$

Consider the group $SO(4)$ and its Lie algebra $so(4)$ paired with itself, via the customary inner product $\langle X, Y \rangle = -\frac{1}{2} \quad tr \quad (X.Y)$, where

$$X = \begin{pmatrix} 0 & -x_3 & x_2 & -x_4 \\ x_3 & 0 & -x_1 & -x_5 \\ -x_2 & x_1 & 0 & -x_6 \\ x_4 & x_5 & x_6 & 0 \end{pmatrix} \in so(4).$$

A left invariant metric on $SO(4)$ is defined by a non-singular symmetric linear map $\Lambda$: $so(4) \longrightarrow so(4)$, $X \longmapsto \Lambda.X$, and by the following inner product; given two vectors $gX$ and $gY$ in the tangent space $SO(4)$ at the point $g \in SO(4)$, $\langle gX, gY \rangle = \left\langle X, \Lambda^{-1}.Y \right\rangle$. Then the geodesic flow for this metric takes the following commutator form (Euler-Arnold equations):

$$\dot{X} = [X, \Lambda.X], \tag{9}$$

where

$$\Lambda.X = \begin{pmatrix} 0 & -\lambda_3 x_3 & \lambda_2 x_2 & -\lambda_4 x_4 \\ \lambda_3 x_3 & 0 & -\lambda_1 x_1 & -\lambda_5 x_5 \\ -\lambda_2 x_2 & \lambda_1 x_1 & 0 & -\lambda_6 x_6 \\ \lambda_4 x_4 & \lambda_5 x_5 & \lambda_6 x_6 & 0 \end{pmatrix} \in so(4).$$

In view of the isomorphism between $\left(\mathbb{R}^6, \wedge\right)$, and $(so(4), [,])$ we write the system (9) as

$$\begin{aligned} \dot{x}_1 &= (\lambda_3 - \lambda_2) x_2 x_3 + (\lambda_6 - \lambda_5) x_5 x_6, & \dot{x}_2 &= (\lambda_1 - \lambda_3) x_1 x_3 + (\lambda_4 - \lambda_4) x_4 x_6, \\ \dot{x}_3 &= (\lambda_2 - \lambda_1) x_1 x_2 + (\lambda_5 - \lambda_4) x_4 x_5, & \dot{x}_4 &= (\lambda_3 - \lambda_5) x_3 x_5 + (\lambda_6 - \lambda_2) x_2 x_6, \\ \dot{x}_5 &= (\lambda_4 - \lambda_3) x_3 x_4 + (\lambda_1 - \lambda_6) x_1 x_6, & \dot{x}_6 &= (\lambda_2 - \lambda_4) x_2 x_4 + (\lambda_5 - \lambda_1) x_1 x_5. \end{aligned}$$

These equations can be written as a hamiltonian vector field

$$\dot{x}(t) = J \frac{\partial H}{\partial x}, \quad x \in \mathbb{R}^6, \tag{10}$$

with

$$H = \frac{1}{2} \langle X, \Lambda X \rangle = \frac{1}{2} \left( \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_6 x_6^2 \right),$$

the hamiltonian and

$$J=\begin{pmatrix} 0 & -x_3 & x_2 & 0 & -x_6 & x_5 \\ x_3 & 0 & -x_1 & x_6 & 0 & -x_4 \\ -x_2 & x_1 & 0 & -x_5 & x_4 & 0 \\ 0 & -x_6 & x_5 & 0 & -x_3 & x_2 \\ x_6 & 0 & -x_4 & x_3 & 0 & -x_1 \\ -x_5 & x_4 & 0 & -x_2 & x_1 & 0 \end{pmatrix} \in so(6).$$

We have $\det J=0$, so $m=2n+k$ and $m-k=rk\ J$. Here $m=6$ and $rg\ J=4$, then $n=k=2$. The system (10) has beside the energy $H_1=H$, two trivial constants of motion:

$$H_2=\frac{1}{2}\left(x_1^2+x_2^2+\cdots+x_6^2\right), \quad H_3=x_1x_4+x_2x_5+x_3x_6.$$

Recall that $H_2$ and $H_3$ are called trivial invariants (or Casimir functions) because $J\dfrac{\partial H_2}{\partial x}=J\dfrac{\partial H_3}{\partial x}=0$. In order that the hamiltonian system (10) be completely integrable, it is suffices to have one more integral, which we take of the form

$$H_4=\frac{1}{2}\left(\mu_1x_1^2+\mu_2x_2^2+\cdots+\mu_6x_6^2\right).$$

The four invariants must be functionally independent and in involution, so in particular

$$\{H_4,H_3\}=\left\langle\frac{\partial H_4}{\partial x}, J\frac{\partial H_3}{\partial x}\right\rangle=0,$$

i.e.,

$$\begin{aligned}&\left(\left(\lambda_3-\lambda_2\right)\mu_1+\left(\lambda_1-\lambda_3\right)\mu_2+\left(\lambda_2-\lambda_1\right)\mu_3\right)x_1x_2x_3\\ &+\left(\left(\lambda_6-\lambda_5\right)\mu_1+\left(\lambda_1-\lambda_6\right)\mu_5+\left(\lambda_5-\lambda_1\right)\mu_6\right)x_1x_5x_6\\ &+\left(\left(\lambda_4-\lambda_6\right)\mu_2+\left(\lambda_6-\lambda_2\right)\mu_4+\left(\lambda_2-\lambda_4\right)\mu_6\right)x_2x_4x_6\\ &+\left(\left(\lambda_5-\lambda_4\right)\mu_3+\left(\lambda_3-\lambda_5\right)\mu_4+\left(\lambda_4-\lambda_3\right)\mu_5\right)x_3x_4x_5=0.\end{aligned}$$

Then

$$\left(\lambda_3-\lambda_2\right)\mu_1+\left(\lambda_1-\lambda_3\right)\mu_2+\left(\lambda_2-\lambda_1\right)\mu_3=0, \quad \left(\lambda_6-\lambda_5\right)\mu_1+\left(\lambda_1-\lambda_6\right)\mu_5+\left(\lambda_5-\lambda_1\right)\mu_6=0,$$

$$\left(\lambda_4-\lambda_6\right)\mu_2+\left(\lambda_6-\lambda_2\right)\mu_4+\left(\lambda_2-\lambda_4\right)\mu_6=0, \quad \left(\lambda_5-\lambda_4\right)\mu_3+\left(\lambda_3-\lambda_5\right)\mu_4+\left(\lambda_4-\lambda_3\right)\mu_5=0.$$

Put

$$\mathcal{A}=\begin{pmatrix} \lambda_3-\lambda_2 & \lambda_1-\lambda_3 & \lambda_2-\lambda_1 & 0 & 0 & 0 \\ \lambda_6-\lambda_5 & 0 & 0 & 0 & \lambda_1-\lambda_6 & \lambda_5-\lambda_1 \\ 0 & \lambda_4-\lambda_6 & 0 & \lambda_6-\lambda_2 & 0 & \lambda_2-\lambda_4 \\ 0 & 0 & \lambda_5-\lambda_4 & \lambda_3-\lambda_5 & \lambda_4-\lambda_3 & 0 \end{pmatrix}.$$

The number of solutions of this system is equal to the number of columns of the matrix $\mathcal{A}$ minus the rank of $\mathcal{A}$. If $rk\mathcal{A}=4$, we have two solutions: $\mu_i=1$ lead to the invariant $H_2$ and $\mu_i=\lambda_i$ lead to the invariant

$H_3$. This is unacceptable. If $rk\mathcal{A}=3$, each four-order minor of $\mathcal{A}$ is singular. Now

$$\begin{pmatrix} \lambda_3-\lambda_2 & \lambda_1-\lambda_3 & \lambda_2-\lambda_1 & 0 \\ \lambda_6-\lambda_5 & 0 & 0 & 0 \\ 0 & \lambda_4-\lambda_6 & 0 & \lambda_6-\lambda_2 \\ 0 & 0 & \lambda_5-\lambda_4 & \lambda_3-\lambda_5 \end{pmatrix} = -\left(\lambda_6-\lambda_5\right)C,$$

$$\begin{pmatrix} \lambda_1-\lambda_3 & \lambda_2-\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1-\lambda_6 \\ \lambda_4-\lambda_6 & 0 & \lambda_6-\lambda_2 & 0 \\ 0 & \lambda_5-\lambda_4 & \lambda_3-\lambda_5 & \lambda_4-\lambda_3 \end{pmatrix} = \left(\lambda_1-\lambda_6\right)C,$$

$$\begin{pmatrix} \lambda_2-\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1-\lambda_6 & \lambda_5-\lambda_1 \\ 0 & \lambda_6-\lambda_2 & 0 & \lambda_2-\lambda_4 \\ \lambda_5-\lambda_4 & \lambda_3-\lambda_5 & \lambda_4-\lambda_3 & 0 \end{pmatrix} = -\left(\lambda_2-\lambda_1\right)C,$$

where

$$C \equiv \lambda_1\lambda_6\lambda_4+\lambda_1\lambda_2\lambda_5-\lambda_1\lambda_2\lambda_4+\lambda_3\lambda_6\lambda_5-\lambda_3\lambda_6\lambda_4-\lambda_3\lambda_2\lambda_5$$
$$+\lambda_4\lambda_2\lambda_5+\lambda_4\lambda_1\lambda_3-\lambda_4\lambda_1\lambda_5+\lambda_6\lambda_2\lambda_3-\lambda_6\lambda_2\lambda_5-\lambda_1\lambda_6\lambda_3,$$

and it follows that the condition for which these minors are zero is $C=0$. Notice that this relation holds by cycling the indices: $1\curvearrowright 4, 2\curvearrowright 5, \curvearrowright 3\curvearrowright 6$. Under Manakov [20] conditions,

$$\lambda_1=\frac{\beta_2-\beta_3}{\alpha_2-\alpha_3},\quad \lambda_2=\frac{\beta_1-\beta_3}{\alpha_1-\alpha_3},\quad \lambda_3=\frac{\beta_1-\beta_2}{\alpha_1-\alpha_2},$$
$$\lambda_4=\frac{\beta_1-\beta_4}{\alpha_1-\alpha_4},\quad \lambda_5=\frac{\beta_2-\beta_4}{\alpha_2-\alpha_4},\quad \lambda_6=\frac{\beta_3-\beta_4}{\alpha_3-\alpha_4}, \tag{11}$$

where $\alpha_i,\beta_i\in\mathbb{C}$, $\prod_{i<j}(\alpha_i-\beta_j)\neq 0$, equations (10) admits a Lax equation with an indeterminate $h$:

$$\overset{\cdot}{\overbrace{(X+\alpha h)}}=\left[X+\alpha h,\Lambda X+\beta h\right], \tag{12}$$

$$\alpha=\begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & \alpha_4 \end{pmatrix},\quad \beta=\begin{pmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \\ 0 & 0 & \beta_3 & 0 \\ 0 & 0 & 0 & \beta_4 \end{pmatrix}.$$

$$\Updownarrow$$

$$\dot{X}=\left[X,\Lambda.X\right]\Leftrightarrow(9),$$
$$\left[X,\beta\right]+\left[\alpha,\Lambda.X\right]=0\Leftrightarrow(11),$$
$$\left[\alpha,\beta\right]=0 \text{ trivially satisfied for diagonal matrices.}$$

The parameters $\mu_1,\ldots,\mu_6$ can be parameterized (like $\lambda_1,\ldots,\lambda_6$) by:

$$\mu_1=\frac{\gamma_2-\gamma_3}{\alpha_2-\alpha_3},\quad \mu_2=\frac{\gamma_1-\gamma_3}{\alpha_1-\alpha_3},\quad \mu_3=\frac{\gamma_1-\gamma_2}{\alpha_1-\alpha_2},\quad \mu_4=\frac{\gamma_1-\gamma_4}{\alpha_1-\alpha_4},\quad \mu_5=\frac{\gamma_2-\gamma_4}{\alpha_2-\alpha_4},\quad \mu_6=\frac{\gamma_3-\gamma_4}{\alpha_3-\alpha_4}.$$

To use the method of isospectral deformations, consider the Kac-Moody extension ($n = 4$): $\mathcal{L} = \left\{\sum_{-\infty}^{N} A_i h^i : N\text{arbitrary} \in \mathbb{Z}, A_i \in gl(n,\mathbb{R})\right\}$, of $gl(n,\mathbb{R})$ with the bracket: $\left[\sum A_i h^i, \sum B_j h^j\right] = \sum_k \left(\sum_{i+j=k} [A_i, B_j]\right) h^k$, and the ad-invariant form: $\left\langle \sum A_i h^i, \sum B_j h^j \right\rangle = \sum_{i+j=-1} \langle A_i, B_j\rangle$, where $\langle , \rangle$ is the usual form defined on $gl(n,\mathbb{R})$. Let $\mathcal{K}$ and $\mathcal{N}$ be respectively the $\geq 0$ and $< 0$ powers of $h$ in $\mathcal{L}$, then $\mathcal{L} = \mathcal{K} + \mathcal{N}$, for the pairing defined above $\mathcal{K} = \mathcal{K}^{\perp}$, $\mathcal{N} = \mathcal{N}^{\perp}$, so that $\mathcal{K} = \mathcal{N}^*$. The orbits described in this way come equipped with a symplectic structure with Poisson bracket $\{H_1, H_2\}(\alpha) = \langle \alpha, [\nabla_{\mathcal{K}^*} H_1, \nabla_{\mathcal{K}^*} H_2]\rangle$, where $\alpha \in \mathcal{K}^*$ and $\nabla_{\mathcal{K}^*} H \in \mathcal{K}$. According to the Adler-Kostant-Symes theorem, the flow (12) is hamiltonian on an orbit through the point $X + ah$, $X \in so(4)$ formed by the coadjoint action of the subgroup $G_{\mathcal{N}} \subset SL(n)$ of lower triangular matrices on the dual Kac-Moody algebra $\mathcal{N}^* \approx \mathcal{K}^{\perp} = \mathcal{K}$. As a consequence, the coefficients of $z^i h^i$ appearing in curve:

$$\Gamma : \quad \left\{(z,h) \in \mathbb{C}^2 : \det\left(X + ah - zI\right) = 0\right\}, \tag{13}$$

associated to the equation (12), are invariant of the system in involution for the symplectic structure of this orbit. Notice that

$$\det\left(gXg^{-1}\right) = \det X = \left(x_1x_4 + x_2x_5 + x_3x_6\right)^2,$$

$$tr\left(gXg^{-1}\right)^2 = tr\left(gX^2g^{-1}\right) = tr\left(X^2\right) = -2\left(x_1^2 + x_2^2 + \cdots + x_6^2\right).$$

Also the complex flows generated by these invariants can be realized as straight lines on the abelian variety defined by the periods of curve $\Gamma$. Explicitly, equation (13) looks as follows

$$\Gamma : \quad \prod_{i=1}^{4}(\alpha_i h - z) + 2H_4h^2 - 2H_1zh + 2H_2z^2 + H_3^2 = 0,$$

where $H_1(X) = c_1$, $H_2(X) = c_2$, $H_3(X) = 2H = c_3$, $H_4(X) = c_4$, with $c_1, c_2, c_3, c_4$ generic constants. $\Gamma$ is a curve of genus 3 and it has a natural involution $\sigma$: $\Gamma \to \Gamma$, $(z,h) \mapsto (-z,-h)$. Therefore the jacobian variety $Jac(\Gamma)$ of $\Gamma$ splits up into an even and old part: the even part is an elliptic curve $\Gamma_0 = \Gamma/\sigma$ and the odd part is a 2-dimensional abelian surface $Prym(\Gamma/\Gamma_0)$ called the Prym variety: $Jac(\Gamma) = \Gamma_0 + Prym(\Gamma/\Gamma_0)$. The van Moerbeke-Mumford linearization method provides then an algebraic map from the complex affine variety $\bigcap_{i=1}^{4}\{H_i(X) = c_i\} \subset \mathbb{C}^6$ to the Jacobi variety $Jac(\Gamma)$. By the antisymmetry of $\Gamma$, this map sends this variety to the Prym variety $Prym(\Gamma/\Gamma_0)$:

$$\bigcap_{i=1}^{4}\{H_i(X) = c_i\} \to Prym(\Gamma/\Gamma_0), \quad p \mapsto \sum_{k=1}^{3} s_k,$$

and the complex flows generated by the constants of the motion are straight lines on $Prym(\Gamma/\Gamma_0)$. Finally, we have the

**Theorem 8.** *The geodesic flow* (9) *is a hamiltonian system with*

$$H \equiv H_1 = \frac{1}{2}\left(\lambda_1x_1^2 + \lambda_2x_2^2 + \cdots + \lambda_6x_6^2\right),$$

*the hamiltonian. It has two trivial invariants*

$$H_2 = \frac{1}{2}\left(x_1^2 + x_2^2 + \cdots + x_6^2\right), \quad H_3 = x_1x_4 + x_2x_5 + x_3x_6.$$

*Moreover, if*

$$\lambda_1\lambda_6\lambda_4 + \lambda_1\lambda_2\lambda_5 - \lambda_1\lambda_2\lambda_4 + \lambda_3\lambda_6\lambda_5 - \lambda_3\lambda_6\lambda_4 - \lambda_3\lambda_2\lambda_5$$
$$+\lambda_4\lambda_2\lambda_5 + \lambda_4\lambda_1\lambda_3 - \lambda_4\lambda_1\lambda_5 + \lambda_6\lambda_2\lambda_3 - \lambda_6\lambda_2\lambda_5 - \lambda_1\lambda_6\lambda_3 = 0,$$

*the system (9) has a fourth independent constant of the motion of the form*

$$H_4 = \frac{1}{2}\left(\mu_1x_1^2 + \mu_2x_2^2 + \cdots + \mu_6x_6^2\right).$$

*Then the system (9) is completely integrable and can be linearized on the Prym variety* $Prym(\Gamma/\Gamma_0)$.

### 2.3. The Toda Lattice

The Toda lattice equations (discretized version of the Korteweg-de Vries equation[3] ) motion of $n$ particles with exponential restoring forces are governed by the following hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \sum_{i=1}^{N} e^{q_i - q_{i+1}}, \qquad q_{N+1} = q_1.$$

The hamiltonian equations can be written as follows

$$\dot{q}_i = p_i, \quad \dot{p}_i = -e^{q_i - q_{i+1}} + e^{q_{i-1} - q_i}.$$

In term of the Flaschka's variables [6]: $a_i = \frac{1}{2}e^{q_i - q_{i+1}}$, $b_i = -\frac{1}{2}p_i$, Toda's equations take the following form

$$\dot{a}_i = a_i\left(b_{i+1} - b_i\right), \quad \dot{b}_i = 2(a_i^2 - a_{i-1}^2), \tag{14}$$

with $b_{N+1} = b_1$ and $a_0 = a_N$. To show that the system (14) is completely integrable, one should find $N$ first integrals independent and in involution each other. From the second equation, we have

$$\frac{d}{dt}\sum_{i=1}^{N} b_i = \sum_{i=1}^{N} \frac{db_i}{dt} = 0,$$

and we normalize the $b_i$'s by requiring that $\sum_{i=1}^{N} b_i = 0$. This is a first integral for the system. We further define $N \times N$ matrices $A$ and $B$ with

$$A = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & a_N \\ a_1 & b_2 & \vdots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & b_{N-1} & a_{N-1} \\ a_N & \cdots & 0 & a_{N-1} & b_N \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_1 & \cdots & \cdots & -a_N \\ -a_1 & 0 & \vdots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{N-1} \\ a_N & \cdots & \cdots & -a_{N-1} & 0 \end{pmatrix}.$$

Then (14) is equivalent to the Lax equation

$$\dot{A} = [B, A].$$

From Theorem 3, we know that the quantities $I_k = \dfrac{1}{k} trA^k$, $1 \leq k \leq N$, are first integrals of motion: To be more precise

$$\dot{I}_k = tr(\dot{A}.A^{k-1}) = tr([B, A].A^{k-1}) = tr(BA^k - ABA^{k-1}) = 0.$$

Notice that $I_1$ is the first integral already know. Since these $N$ first integrals are shown to be independent and in involution each other, the system (14) is thus completely integrable.

### 2.4. The Garnier Potential

Consider the hamiltonian

$$H = \frac{1}{2}\left(x_1^2 + x_2^2\right) - \frac{1}{2}\left(\lambda_1 y_1^2 + \lambda_2 y_2^2\right) + \frac{1}{4}\left(y_1^2 + y_2^2\right)^2, \tag{15}$$

where $\lambda_1$ and $\lambda_2$ are constants. The corresponding system is given by

$$\begin{aligned} \dot{y}_1 &= x_1, & \dot{x}_1 &= \left(\lambda_1 - y_1^2 - y_2^2\right) y_1, \\ \dot{y}_2 &= x_2, & \dot{x}_2 &= \left(\lambda_2 - y_2^2 - y_1^2\right) y_2. \end{aligned} \tag{16}$$

[3] In short K-dV equation: $\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$. This is an infinite-dimensional completely integrable system.

**Theorem 9.** *The system* (16) *has the additional first integral*

$$H_2 = \frac{1}{4}\left((x_1y_2 - x_2y_1)^2 - \left(\lambda_2 y_1^4 + \lambda_1 y_2^4\right) - (\lambda_1+\lambda_2)\, y_1^2y_2^2\right) + \frac{1}{2}\left(\lambda_1\lambda_2\left(y_1^2+y_2^2\right) - \left(\lambda_2 x_1^2 + \lambda_1 x_2^2\right)\right)$$

*and is completely integrable. The flows generated by $H_1 = H$(15) and $H_2$ are straight line motions on the jacobian variety of a smooth genus two hyperelliptic curve $\mathcal{H}$(17) associated to a Lax equation.*

**Proof.** We consider the Lax representation in the form $\dot{A} = [A, B]$, with the following ansatz for the Lax operator

$$A = \begin{pmatrix} U & V \\ W & -U \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ R & 0 \end{pmatrix}$$

where

$$V = -(h-\lambda_1)(h-\lambda_2)\left(1 + \frac{1}{2}\left(\frac{y_1^2}{h-\lambda_1} + \frac{y_2^2}{h-\lambda_2}\right)\right),$$

$$U = \frac{1}{2}(h-\lambda_1)(h-\lambda_2)\left(\frac{x_1y_1}{h-\lambda_1} + \frac{x_2y_2}{h-\lambda_2}\right),$$

$$W = (h-\lambda_1)(h-\lambda_2)\left(\frac{1}{2}\left(\frac{x_1^2}{h-\lambda_1} + \frac{x_2^2}{h-\lambda_2}\right) - h + \frac{1}{2}(y_1^2+y_2^2)\right),$$

$$R = h - y_1^2 - y_2^2.$$

We form the curve in $(z, h)$ space

$$P(h,z) = \det(A - zI) = 0,$$

whose coefficients are functions of the phase space. Explicitly, this equation looks as follows

$$\mathcal{H}: \quad z^2 = P_5(h) \tag{17}$$
$$= (h-\lambda_1)(h-\lambda_2)(h^3 - (\lambda_1+\lambda_2)h^2 + (\lambda_1\lambda_2 - H_1)h - H_2),$$

with $H_1$ (15) the hamiltonian and a second quartic integral $H_2$ of the form

$$H_2 = -\frac{1}{4}(\lambda_2 y_1^4 + \lambda_1 y_2^4 + (\lambda_1+\lambda_2)y_1^2y_2^2 - (x_1y_2 - x_2y_1)^2) - \frac{1}{2}(\lambda_2 x_1^2 + \lambda_1 x_2^2 - \lambda_1\lambda_2(y_1^2+y_2^2)).$$

The functions $H_1$ and $H_2$ commute: $\{H_1, H_2\} = 0$ and the system (16) is completely integrable. The curve $\mathcal{H}$ determined by the fifth-order equation (17) is smooth, hyperelliptic and its genus is 2. Obviously, $\mathcal{H}$ is invariant under the hyperelliptic involution $(h, z) \curvearrowright (h, -z)$. Using the van Moerbeke-Mumford linearization method, we show that the linearized flow could be realized on the jacobian variety $Jac(\mathcal{H})$ of the genus 2 curve $\mathcal{H}$. For generic $c = (c_1, c_2) \in \mathbb{C}^2$ the affine variety defined by

$$M_c = \bigcap_{i=1}^{2}\left\{x \in \mathbb{C}^4 : H_i(x) = c_i\right\},$$

is a smooth affine surface. According to the schema of [5], we introduce coordinates $s_1$ and $s_2$ on the surface $M_c$, such that $M_c(s_i) = 0, \lambda_1 \neq \lambda_2$, i.e.,

$$s_1 + s_2 = \frac{1}{2}\left(y_1^2 + y_2^2\right) + \lambda_1 + \lambda_2, \quad s_1s_2 = \frac{1}{2}\left(\lambda_2 y_1^2 + \lambda_1 y_2^2\right) + \lambda_1\lambda_2.$$

After some algebraic manipulations, we obtain the following equations for $s_1$ and $s_2$:

$$\dot{s}_1 = 2\frac{\sqrt{P_5(s_1)}}{s_1 - s_2}, \quad \dot{s}_2 = 2\frac{\sqrt{P_5(s_2)}}{s_2 - s_1},$$

where $P_5(s)$ is defined by (17). These equations can be integrated by the abelian mapping

$$\mathcal{H} \longrightarrow Jac(\mathcal{H}) = \mathbb{C}^2/L, \quad p \longmapsto \left( \int_{p_0}^{p} \omega_1, \quad \int_{p_0}^{p} \omega_2 \right),$$

where the hyperelliptic curve $\mathcal{H}$ of genus two is given by the equation (17), $L$ is the lattice generated by the vectors $n_1 + \Omega n_2, (n_1, n_2) \in \mathbb{Z}^2, \Omega$ is the matrix of period of the curve $\mathcal{H}$, $(\omega_1, \omega_2)$ is a canonical basis of holomorphic differentials on $\mathcal{H}$, i.e.,

$$\omega_1 = \frac{ds}{\sqrt{P_5(s)}}, \quad \omega_2 = \frac{sds}{\sqrt{P_5(s)}},$$

and $p_0$ is a fixed point. This concludes the proof of the theorem.

### *2.5. The Coupled Nonlinear Schrödinger Equations*

The system of two coupled nonlinear Schrödinger equations is given by

$$\begin{aligned} i\frac{\partial a}{\partial z} + \frac{\partial^2 a}{\partial t^2} + \Omega_0 a + \frac{2}{3}\left(|a|^2 + |b|^2\right) a + \frac{1}{3}\left(a^2 + b^2\right) \overline{a} = 0, \\ i\frac{\partial b}{\partial z} + \frac{\partial^2 b}{\partial t^2} - \Omega_0 b + \frac{2}{3}\left(|a|^2 + |b|^2\right) b + \frac{1}{3}\left(a^2 + b^2\right) \overline{b} = 0, \end{aligned} \tag{18}$$

where $a(z,t)$ and $b(z,t)$ are functions of $z$ and $t$, the bar "$-$" denotes the complex conjugation, "$||$" denotes the modulus and $\Omega_0$ is a constant. These equations play a significant role in mathematics, with a important number of physical applications. We seek solutions of (18) in the following form

$$a(z,t) = y_1(t) \exp(i\Omega z), \quad b(z,t) = y_2(t) \exp(i\Omega z),$$

where $y_1(t)$ et $y_2(t)$ are two functions and $\Omega$ is an arbitrary constant. Then we obtain the system

$$\ddot{y}_1 + \left(y_1^2 + y_2^2\right) y_1 = (\Omega - \Omega_0) y_1, \quad \ddot{y}_2 + \left(y_1^2 + y_2^2\right) y_2 = (\Omega + \Omega_0) y_2.$$

The latter coincides obviously with (16) for $\lambda_1 = \Omega - \Omega_0$ and $\lambda_2 = \Omega + \Omega_0$.

### *2.6. The Yang-Mills Equations*

We consider the Yang-Mills system for a field with gauge group $SU(2)$:

$$\nabla_j F_{jk} = \frac{\partial F_{jk}}{\partial \tau_j} + [A_j, F_{jk}] = 0,$$

where $F_{jk}, A_j \in T_e SU(2), 1 \leq j,k \leq 4$ and $F_{jk} = \dfrac{\partial A_k}{\partial \tau_j} - \dfrac{\partial A_j}{\partial \tau_k} + [A_j, A_k]$. The self-dual Yang-Mills (SDYM) equations is an universal system for which some reductions include all classical tops from Euler to Kowalewski ($0+1$-dimensions), K-dV, Nonlinear Schrödinger, Sine-Gordon, Toda lattice and N-waves equations ($1+1$-dimensions), KP and D-S equations ($2+1$-dimensions). In the case of homogeneous double-component field, we have $\partial_j A_k = 0, j \neq 1, A_1 = A_2 = 0, A_3 = n_1 U_1 \in su(2), A_4 = n_2 U_2 \in su(2)$ where $n_i$ are $su(2)$-generators (i.e., they satisfy commutation relations: $n_1 = [n_2, [n_1, n_2]], n_2 = [n_1, [n_2, n_1]]$). The system becomes

$$\frac{\partial^2 U_1}{\partial t^2} + U_1 U_2^2 = 0, \qquad \frac{\partial^2 U_2}{\partial t^2} + U_2 U_1^2 = 0,$$

with $t = \tau_1$. By setting $U_j = q_j$, $\dfrac{\partial U_j}{\partial t} = p_j$, $j = 1,2$, Yang-Mills equations are reduced to hamiltonian system

$$\dot{x} = J\frac{\partial H}{\partial x}, \quad x = (q_1, q_2, p_1, p_2)^\intercal, \quad J = \begin{pmatrix} O & -I \\ I & O \end{pmatrix},$$

with $H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 q_2^2)$, the hamiltonian. The symplectic transformation $p_1 \curvearrowleft \frac{\sqrt{2}}{2}(p_1 + p_2), p_2 \curvearrowleft \frac{\sqrt{2}}{2}(p_1 - p_2), q_1 \curvearrowleft \frac{1}{2}(\sqrt[4]{2})(q_1 + iq_2), q_2 \curvearrowleft \frac{1}{2}(\sqrt[4]{2})(q_1 - iq_2)$, takes this hamiltonian into

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{4}q_1^4 + \frac{1}{4}q_2^4 + \frac{1}{2}q_1^2 q_2^2,$$

which coincides with (15) for $\lambda_1 = \lambda_2 = 0$.

## 3. ALGEBRAIC COMPLETE INTEGRABILITY

We give some results about abelian surfaces which will be used, as well as the basic techniques to study two-dimensional algebraic completely integrable systems. Let $M = \mathbb{C}/\Lambda$ be a $n$-dimensional abelian variety where $\Lambda$ is the lattice generated by the $2n$ columns $\lambda_1, \ldots, \lambda_{2n}$ of the $n \times 2n$ period matrix $\Omega$ and let $D$ be a divisor on $M$. Define $\mathcal{L}(\mathcal{D}) = \{f \text{meromorphic on} M : (f) \geq -\mathcal{D}\}$, i.e., for $\mathcal{D} = \sum k_j \mathcal{D}_j$ a function $f \in \mathcal{L}(\mathcal{D})$ has at worst a $k_j$-fold pole along $\mathcal{D}_j$. The divisor $\mathcal{D}$ is called ample when a basis $(f_0, \ldots, f_N)$ of $\mathcal{L}(k\mathcal{D})$ embeds $M$ smoothly into $\mathbb{P}^N$ for some $k$, via the map $M \to \mathbb{P}^N$, $p \mapsto [1 : f_1(p) : \ldots : f_N(p)]$, then $k\mathcal{D}$ is called very ample. It is known that every positive divisor $\mathcal{D}$ on an irreducible abelian variety is ample and thus some multiple of $\mathcal{D}$ embeds $M$ into $\mathbb{P}^N$. By a theorem of Lefschetz, any $k \geq 3$ will work. Moreover, there exists a complex basis of $\mathbb{C}^n$ such that the lattice expressed in that basis is generated by the columns of the $n \times 2n$ period matrix

$$\begin{pmatrix} \delta_1 & & 0 & | & \\ & \ddots & & | & Z \\ 0 & & \delta_n & | & \end{pmatrix},$$

with $Z^\top = Z, \mathrm{Im} Z > 0, \delta_j \in \mathbb{N}^*$ and $\delta_j | \delta_{j+1}$. The integers $\delta_j$ which provide the so-called polarization of the abelian variety $M$ are then related to the divisor as follows:

$$\dim \mathcal{L}(\mathcal{D}) = \delta_1 \ldots \delta_n. \tag{19}$$

In the case of a 2-dimensional abelian varieties (surfaces), even more can be stated: the geometric genus $g$ of a positive divisor $\mathcal{D}$ (containing possibly one or several curves) on a surface $M$ is given by the adjunction formula

$$g(\mathcal{D}) = \frac{K_M.\mathcal{D} + \mathcal{D}.\mathcal{D}}{2} + 1, \tag{20}$$

where $K_M$ is the canonical divisor on $M$, i.e., the zero-locus of a holomorphic 2-form, $\mathcal{D}.\mathcal{D}$ denote the number of intersection points of $\mathcal{D}$ with $a + \mathcal{D}$ (where $a + \mathcal{D}$ is a small translation by $a$ of $\mathcal{D}$ on $M$), where as the Riemann-Roch theorem for line bundles on a surface tells you that

$$\chi(\mathcal{D}) = p_a(M) + 1 + \frac{1}{2}(\mathcal{D}.\mathcal{D} - \mathcal{D}K_M), \tag{21}$$

where $p_a(M)$ is the arithmetic genus of $M$ and $\chi(\mathcal{D})$ the Euler characteristic of $\mathcal{D}$. To study abelian surfaces using Riemann surfaces on these surfaces, we recall that

$$\begin{aligned} \chi(\mathcal{D}) &= \dim H^0(M, \mathcal{O}_M(\mathcal{D})) - \dim H^1(M, \mathcal{O}_M(D)) \qquad (22) \\ &= \dim \mathcal{L}(\mathcal{D}) - \dim H^1(M, \Omega^2(\mathcal{D} \otimes K_M^*)), (\text{Kodaira-Serre duality}) \\ &= \dim \mathcal{L}(\mathcal{D}), (\text{Kodaira vanishing theorem}), \end{aligned}$$

whenever $\mathcal{D} \otimes K_M^*$ defines a positive line bundle. However for abelian surfaces, $K_M$ is trivial and $p_a(M) = -1$; therefore combining relations (19), (20), (21) and (22),

$$\chi(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}) = \frac{\mathcal{D}.\mathcal{D}}{2} = g(\mathcal{D}) - 1 = \delta_1 \delta_2.$$

A divisor $\mathcal{D}$ is called projectively normal, when the natural map $\mathcal{L}(\mathcal{D})^{\otimes k} \to \mathcal{L}(k\mathcal{D})$, is surjective, i.e., every function of $\mathcal{L}(k\mathcal{D})$ can be written as a linear combination of k-fold products of functions of $\mathcal{L}(\mathcal{D})$. Not every very ample divisor $\mathcal{D}$ is projectively normal but if $\mathcal{D}$ is linearly equivalent to $k\mathcal{D}_0$ for $k \geq 3$ for some divisor $\mathcal{D}_0$, then $\mathcal{D}$ is projectively normal.

Now consider the exact sheaf sequence

$$0 \longrightarrow \mathcal{O}_C \overset{\pi^*}{\longrightarrow} \mathcal{O}_{\widetilde{C}} \longrightarrow X \longrightarrow 0,$$

where $C$ is a singular connected Riemann surface, $\widetilde{C} = \sum C_j$ the corresponding set of smooth Riemann surfaces after desingularization and $\pi\colon \widetilde{C} \to C$ the projection. The exactness of the sheaf sequence shows that the Euler characteristic

$$\mathcal{X}(\mathcal{O}) = \dim H^0(\mathcal{O}) - \dim H^1(\mathcal{O}),$$

satisfy

$$\mathcal{X}(\mathcal{O}_C) - \mathcal{X}(\mathcal{O}_{\widetilde{C}}) + \mathcal{X}(X) = 0, \tag{23}$$

where $\mathcal{X}(X)$ only accounts for the singular points $p$ of $C$; $\mathcal{X}(X_p)$ is the dimension of the set of holomorphic functions on the different branches around p taken separately, modulo the holomorphic functions on the Riemann surface $C$ near that singular point. Consider the case of a planar singularity (in this paper, we will be concerned by a tacnode for which $\mathcal{X}(X) = 2$, as well), i.e., the tangents to the branches lie in a plane. If $f_j(x, y) = 0$ denote the $j^{th}$ branch of $C$ running through p with local parameter $s_j$, then

$$\mathcal{X}(X_p) = \dim \Pi_j \mathbb{C}[[s_j]] / \frac{\mathbb{C}[[x, y]]}{\Pi_j f_j(x, y)}.$$

So using (22) and Serre duality, we obtain $\mathcal{X}(\mathcal{O}_C) = 1 - g(C)$ and $\mathcal{X}(\mathcal{O}_{\widetilde{C}}) = n - \sum_{j=1}^n g(C_j)$. Also, replacing in the formula (23), gives

$$g(C) = \sum_{j=1}^n g(C_j) + \mathcal{X}(X) + 1 - n.$$

Finally, recall that a Kähler variety is a variety with a Kähler metric, i.e., a hermitian metric whose associated differential 2-form of type $(1, 1)$ is closed. The complex torus $\mathbb{C}^2/lattice$ with the euclidean metric $\sum dz_i \otimes d\overline{z}_i$ is a Kähler variety and any compact complex variety that can be embedded in projective space is also a Kähler variety. Now, a compact complex Kähler variety having as many independent meromorphic functions as its dimension is a projective variety.

Consider now hamiltonian problems of the form

$$X_H : \dot{x} = J\frac{\partial H}{\partial x} \quad \equiv f(x), \quad x \in \mathbb{R}^m, \tag{24}$$

where $H$ is the hamiltonian and $J = J(x)$ is a skew-symmetric matrix with polynomial entries in $x$, for which the corresponding Poisson bracket $\{H_i, H_j\} = \left\langle \frac{\partial H_i}{\partial x}, J\frac{\partial H_j}{\partial x} \right\rangle$, satisfies the Jacobi identities. The system (24) with polynomial right hand side will be called algebraic complete integrable (a.c.i.) when:

(i) The system possesses $n + k$ independent polynomial invariants $H_1, \ldots, H_{n+k}$ of which $k$ lead to zero vector fields $J\frac{\partial H_{n+i}}{\partial x}(x) = 0, 1 \leq i \leq k$, the $n$ remaining ones are in involution (i.e., $\{H_i, H_j\} = 0$) and $m = 2n + k$. For most values of $c_i \in \mathbb{R}$, the invariant varieties $\bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m : H_i = c_i\}$ are assumed compact and connected. Then, according to the Arnold-Liouville theorem, there exists a diffeomorphism

$$\bigcap_{i=1}^{n+k} \{x \in \mathbb{R}^m : H_i = c_i\} \to \mathbb{R}^n/Lattice,$$

and the solutions of the system (24) are straight lines motions on these tori.

(ii) The invariant varieties, thought of as affine varieties in $\mathbb{C}^m$ can be completed into complex algebraic tori, i.e.,

$$\bigcap_{i=1}^{n+k}\{H_i = c_i, x \in \mathbb{C}^m\} \cup \mathcal{D} = \mathbb{C}^n/Lattice,$$

where $\mathbb{C}^n/Lattice$ is a complex algebraic torus (i.e., abelian variety) and $\mathcal{D}$ a divisor. Algebraic means that the torus can be defined as an intersection $\bigcap_{i=1}^{M}\{P_i(X_0, \ldots, X_N) = 0\}$ involving a large number of homogeneous polynomials $P_i$. In the natural coordinates $(t_1, \ldots, t_n)$ of $\mathbb{C}^n/Lattice$ coming from $\mathbb{C}^n$, the functions $x_i = x_i(t_1, \ldots, t_n)$ are meromorphic and (24) defines straight line motion on $\mathbb{C}^n/Lattice$. Condition (i) means, in particular, there is an algebraic map $(x_1(t), \ldots, x_m(t)) \mapsto (\mu_1(t), \ldots, \mu_n(t))$ making the following sums linear in $t$:

$$\sum_{i=1}^{n}\int_{\mu_i(0)}^{\mu_i(t)} \omega_j = d_j t, \quad 1 \le j \le n, \quad d_j \in \mathbb{C},$$

where $\omega_1, \ldots, \omega_n$ denote holomorphic differentials on some algebraic curves.

The existence of a coherent set of Laurent solutions:

$$x_i = \sum_{j=0}^{\infty} x_i^{(j)} t^{j-k_i}, \quad k_i \in \mathbb{Z}, \quad \text{some } k_i > 0, \tag{25}$$

depending on dim (phase space) $-1 = m-1$ free parameters is necessary and sufficient for a hamiltonian system with the right number of constants of motion to be a.c.i. So, if the hamiltonian flow (24) is a.c.i., it means that the variables $x_i$ are meromorphic on the torus $\mathbb{C}^n/Lattice$ and by compactness they must blow up along a codimension one subvariety (a divisor) $\mathcal{D} \subset \mathbb{C}^n/Lattice$. By the a.c.i. definition, the flow (24) is a straight line motion in $\mathbb{C}^n/Lattice$ and thus it must hit the divisor $\mathcal{D}$ in at least one place. Moreover through every point of $\mathcal{D}$, there is a straight line motion and therefore a Laurent expansion around that point of intersection. Hence the differential equations must admit Laurent expansions which depend on the $n-1$ parameters defining $\mathcal{D}$ and the $n+k$ constants $c_i$ defining the torus $\mathbb{C}^n/Lattice$, the total count is therefore $m-1 = dim\ (phase\ space) - 1$ parameters.

Assume now hamiltonian flows to be (weight)-homogeneous with a weight $\nu_i \in \mathbb{N}$, going with each variable $x_i$, i.e.,

$$f_i\left(\alpha^{\nu_1}x_1, \ldots, \alpha^{\nu_m}x_m\right) = \alpha^{\nu_i+1} f_i\left(x_1, \ldots, x_m\right), \quad \forall \alpha \in \mathbb{C}.$$

Observe that then the constants of the motion $H$ can be chosen to be (weight)-homogeneous:

$$H\left(\alpha^{\nu_1}x_1, \ldots, \alpha^{\nu_m}x_m\right) = \alpha^{k} H\left(x_1, \ldots, x_m\right), \quad k \in \mathbb{Z}.$$

If the flow is algebraically completely integrable, the differential equations (24) must admits Laurent series solutions (25) depending on $m-1$ free parameters. We must have $k_i = \nu_i$ and coefficients in the series must satisfy at the 0th step non-linear equations,

$$f_i\left(x_1^{(0)}, \ldots, x_m^{(0)}\right) + g_i x_i^{(0)} = 0, \quad 1 \le i \le m, \tag{26}$$

and at the $k$th step, linear systems of equations:

$$\left(L - kI\right) z^{(k)} = \begin{cases} 0 & \text{for } k = 1 \\ \text{some polynomial in} & x^{(1)}, \ldots, x^{(k-1)} \text{ for } k > 1, \end{cases} \tag{27}$$

where

$$L = \text{ Jacobian map of (26)} = \frac{\partial f}{\partial z} + gI\ |_{z=z^{(0)}}\ .$$

If $m-1$ free parameters are to appear in the Laurent series, they must either come from the non-linear equations (26) or from the eigenvalue problem (27), i.e., $L$ must have at least $m-1$ integer eigenvalues. These are much less conditions than expected, because of the fact that the homogeneity $k$ of the constant $H$ must be an eigenvalue of $L$ Moreover the formal series solutions are convergent as a consequence of the majorant method. Next we assume that the divisor is very ample and in addition projectively normal. Consider a point $p \in \mathcal{D}$, a chart $U_j$ around $p$ on the torus and a function $y_j$ in $\mathcal{L}(\mathcal{D})$ having a pole of maximal order at $p$. Then the vector $(1/y_j, y_1/y_j, \ldots, y_N/y_j)$ provides a good system of coordinates in $U_j$. Then taking the derivative with regard to one of the flows

$$\left(\frac{y_i}{y_j}\right)^{\cdot} = \frac{\dot{y}_i y_j - y_i \dot{y}_j}{y_j^2}, \quad 1 \le j \le N,$$

are finite on $U_j$ as well. Therefore, since $y_j^2$ has a double pole along $\mathcal{D}$, the numerator must also have a double pole (at worst), i.e., $\dot{y}_i y_j - y_i \dot{y}_j \in \mathcal{L}(2\mathcal{D})$. Hence, when $\mathcal{D}$ is projectively normal, we have that

$$\left(\frac{y_i}{y_j}\right)^{\cdot} = \sum_{k,l} a_{k,l} \left(\frac{y_k}{y_j}\right)\left(\frac{y_l}{y_j}\right),$$

i.e., the ratios $y_i/y_j$ form a closed system of coordinates under differentiation. At the bad points, the concept of projective normality play an important role: this enables one to show that $y_i/y_j$ is a bona fide Taylor series starting from every point in a neighbourhood of the point in question.

To prove the algebraic complete integrability of a given hamiltonian system, the main steps of the method are:

—The first step is to show the existence of the Laurent solutions, which requires an argument precisely every time $k$ is an integer eigenvalue of $L$ and therefore $L - kI$ is not invertible.

—One shows the existence of the remaining constants of the motion in involution so as to reach the number $n+k$.

—For given $c_1, \ldots, c_m$, the set

$$\mathcal{D} \equiv \left\{ \begin{array}{l} x_i(t) = t^{-\nu_i}\left(x_i^{(0)} + x_i^{(1)} t + x_i^{(2)} t^2 + \cdots\right), \quad 1 \le i \le m \\ \text{Laurent solutions such that} \ : \quad H_j(x_i(t)) = c_j + \text{ Taylor part }, \end{array} \right\}$$

defines one or several $n-1$ dimentional algebraic varieties (divisor) having the property that

$$\begin{aligned} \bigcap_{i=1}^{n+k} \{H_i = c_i, z \in \mathbb{C}^m\} \cup \mathcal{D} \quad &= \text{a smooth compact, connected variety} \\ &\quad \text{with } n \text{ commuting vector fields} \\ &\quad \text{independent at every point.} \\ &= \text{ a complex algebraic torus } T^n = \mathbb{C}^n/\mathit{Lattice}. \end{aligned}$$

The flows $J\dfrac{\partial H_{k+i}}{\partial z}, \ldots, J\dfrac{\partial H_{k+n}}{\partial z}$ are straight line motions on $T^n$.

From the divisor $\mathcal{D}$, a lot of information can be obtained with regard to the periods and the action-angle variables.

### *3.1. A Five-Dimensional System*

Consider the following system of five differential equations in the unknowns $z_1, \ldots, z_5$:

$$\begin{aligned} \dot{z}_1 &= 2z_4, \qquad \dot{z}_3 = z_2(3z_1 + 8z_2^2), \\ \dot{z}_2 = z_3, \qquad \dot{z}_4 &= z_1^2 + 4z_1 z_2^2 + z_5, \qquad \dot{z}_5 = 2z_1 z_4 + 4z_2^2 z_4 - 2z_1 z_2 z_3. \end{aligned} \tag{28}$$

The following three quartics are constants of motion for this system

$$F_1 = \frac{1}{2} z_5 - z_1 z_2^2 + \frac{1}{2} z_3^2 - \frac{1}{4} z_1^2 - 2z_2^4, \tag{29}$$

$$F_2 = z_5^2 - z_1^2 z_5 + 4z_1z_2z_3z_4 - z_1^2z_3^2 + \frac{1}{4}z_1^4 - 4z_2^2z_4^2, \quad F_3 = z_1z_5 + z_1^2z_2^2 - z_4^2.$$

This system is completely integrable and the hamiltonian structure is defined by the Poisson bracket $\{F, H\} = \left\langle \frac{\partial F}{\partial z}, J\frac{\partial H}{\partial z} \right\rangle = \sum_{k,l=1}^5 J_{kl}\frac{\partial F}{\partial z_k}\frac{\partial H}{\partial z_l}$, where $\frac{\partial H}{\partial z} = \left(\frac{\partial H}{\partial z_1}, \frac{\partial H}{\partial z_2}, \frac{\partial H}{\partial z_3}, \frac{\partial H}{\partial z_4}, \frac{\partial H}{\partial z_5}\right)^\top$, and

$$J = \begin{bmatrix} 0 & 0 & 0 & 2z_1 & 4z_4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -4z_1z_2 \\ -2z_1 & 0 & 0 & 0 & 2z_5 - 8z_1z_2^2 \\ -4z_4 & 0 & 4z_1z_2 & -2z_5 + 8z_1z_2^2 & 0 \end{bmatrix},$$

is a skew-symmetric matrix for which the corresponding Poisson bracket satisfies the Jacobi identities. The system (28) can be written as $\dot{z} = J\frac{\partial H}{\partial z}$, $z = (z_1, z_2, z_3, z_4, z_5)^\top$, where $H = F_1$. The second flow commuting with the first is regulated by the equations $\dot{z} = J\frac{\partial F_2}{\partial z}$, $z = (z_1, z_2, z_3, z_4, z_5)^\top$. These vector fields are in involution: $\{F_1, F_2\} = \left\langle \frac{\partial F_1}{\partial z}, J\frac{\partial F_2}{\partial z} \right\rangle = 0$, and the remaining one is casimir: $J\frac{\partial F_3}{\partial z} = 0$. The invariant variety $A$ defined by

$$A = \bigcap_{k=1}^{2} \{z : F_k(z) = c_k\} \subset \mathbb{C}^5, \tag{30}$$

is a smooth affine surface for generic values of $(c_1, c_2, c_3) \in \mathbb{C}^3$. So, the question I address is how does one find the compactification of $A$ into an abelian surface? The idea of the direct proof we shall give here is closely related to the geometric spirit of the (real) Arnold-Liouville theorem. Namely, a compact complex $n$-dimensional variety on which there exist $n$ holomorphic commuting vector fields which are independent at every point is analytically isomorphic to a $n$-dimensional complex torus $\mathbb{C}^n/Lattice$ and the complex flows generated by the vector fields are straight lines on this complex torus. Now, the main problem will be to complete $A$ (30) into a non singular compact complex algebraic variety $\widetilde{A} = A \cup \mathcal{D}$ in such a way that the vector fields $X_{F_1}$ and $X_{F_2}$ generated respectively by $F_1$ and $F_2$, extend holomorphically along a divisor $\mathcal{D}$ and remain independent there. If this is possible, $\widetilde{A}$ is an algebraic complex torus (an abelian variety) and the coordinates $z_1, \ldots, z_5$ restricted to $A$ are abelian functions. A naive guess would be to take the natural compactification $\overline{A}$ of $A$ by projectivizing the equations: $\overline{A} = \bigcap_{k=1}^3 \{F_k(Z) = c_k Z_0^4\} \subset \mathbb{P}^5$. Indeed, this can never work for a general reason: an abelian variety $\widetilde{A}$ of dimension bigger or equal than two is never a complete intersection, that is it can never be described in some projective space $\mathbb{P}^n$ by $n$-dim $\widetilde{A}$ global polynomial homogeneous equations. In other words, if $A$ is to be the affine part of an abelian surface, $\overline{A}$ must have a singularity somewhere along the locus at infinity $\overline{A} \cap \{Z_0 = 0\}$. In fact, we shall show that the existence of meromorphic solutions to the differential equations (28) depending on 4 free parameters can be used to manufacture the tori, without ever going through the delicate procedure of blowing up and down. Information about the tori can then be gathered from the divisor.

**Theorem 10.** *The system (28) possesses Laurent series solutions which depend on 4 free parameters: $\alpha$, $\beta$, $\gamma$ and $\theta$. These meromorphic solutions restricted to the surface $A$(30) are parameterized by two copies $\mathcal{C}_{-1}$ and $\mathcal{C}_1$ of the same Riemann surface (32) of genus 7.*

**Proof.** The first fact to observe is that if the system is to have Laurent solutions depending on 4 free parameters $\alpha, \beta, \gamma, \theta$, the Laurent decomposition of such asymptotic solutions must have the following form

$$z_1 = \frac{1}{t}\alpha - \frac{1}{2}\alpha^2 + \beta t - \frac{1}{16}\alpha\left(\alpha^3 + 4\beta\right)t^2 + \gamma t^3 + \cdots, \tag{31}$$

$$z_2 = \frac{1}{2t}\varepsilon - \frac{1}{4}\varepsilon\alpha + \frac{1}{8}\varepsilon\alpha^2 t - \frac{1}{32}\varepsilon\left(-\alpha^3 + 12\beta\right)t^2 + \theta t^3 + \cdots,$$
$$z_3 = -\frac{1}{2t^2}\varepsilon + \frac{1}{8}\varepsilon\alpha^2 - \frac{1}{16}\varepsilon\left(-\alpha^3 + 12\beta\right)t + 3\theta t^2 + \cdots,$$
$$z_4 = -\frac{1}{2t^2}\alpha + \frac{1}{2}\beta - \frac{1}{16}\alpha\left(\alpha^3 + 4\beta\right)t + \frac{3}{2}\gamma t^2 + \cdots,$$
$$z_5 = \frac{1}{2t^2}\alpha^2 - \frac{1}{4t}\left(\alpha^3 + 4\beta\right) + \frac{1}{4}\alpha\left(\alpha^3 + 2\beta\right) - \left(\alpha^2\beta - 2\gamma + 4\varepsilon\theta\alpha\right)t + \cdots,$$

with $\varepsilon = \pm 1$. Using the majorant method, we can show that these series are convergent. Substituting the Laurent solutions (31) into (29): $F_1 = c_1$, $F_2 = c_2$ and $F_3 = c_3$, and equating the $t^0$-terms yields

$$F_1 = \frac{7}{64}\alpha^4 - \frac{1}{8}\alpha\beta - \frac{5}{2}\varepsilon\theta = c_1, \quad F_2 = \frac{1}{16}\left(4\beta - \alpha^3\right)\left(4\alpha^2\beta - \alpha^5 + 64\varepsilon\theta\alpha - 32\gamma\right) = c_2,$$
$$F_3 = -\frac{1}{32}\alpha^6 - \beta^2 - \frac{1}{4}\alpha^3\beta - 3\varepsilon\theta\alpha^2 + 4\alpha\gamma = c_3.$$

Eliminating $\gamma$ and $\theta$ from these equations, leads to an equation connecting the two remaining parameters $\alpha$ and $\beta$:

$$\mathcal{C}: \quad 64\beta^3 - 16\alpha^3\beta^2 - 4\left(\alpha^6 - 32\alpha^2 c_1 - 16c_3\right)\beta + \alpha\left(32c_2 - 32\alpha^4 c_1 + \alpha^8 - 16\alpha^2 c_3\right) = 0. \tag{32}$$

The Laurent solutions restricted to the surface $A$ (30) are thus parameterized by two copies $\mathcal{C}_{-1}$ and $\mathcal{C}_1$ of the same Riemann surface $\mathcal{C}$(32). According to the Riemann-Hurwitz formula, the genus of the Riemann surface $\mathcal{C}$ is 7, which establishes the theorem.

In order to embed $\mathcal{C}$ into some projective space, one of the key underlying principles used is the Kodaira embedding theorem, which states that a smooth complex manifold can be smoothly embedded into projective space $\mathbb{P}^N$ with the set of functions having a pole of order $k$ along positive divisor on the manifold, provided $k$ is large enough; fortunately, for abelian varieties, $k$ need not be larger than three according to Lefshetz. These functions are easily constructed from the Laurent solutions (31) by looking for polynomials in the phase variables which in the expansions have at most a $k$-fold pole. The nature of the expansions and some algebraic proprieties of abelian varieties provide a recipe for when to terminate our search for such functions, thus making the procedure implementable. Precisely, we wish to find a set of polynomial functions $\{f_0, \ldots, f_N\}$, of increasing degree in the original variables $z_1, \ldots, z_5$ having the property that the embedding $\mathcal{D}$ of $\mathcal{C}_1 + \mathcal{C}_{-1}$ into $\mathbb{P}^N$ via those functions satisfies the relation: geometric genus $(\mathcal{D}) \equiv g(\mathcal{D}) = N + 2$. A this point, it may be not so clear why $\mathcal{D}$ must really live on an abelian surface. Let us say, for the moment, that the equations of the divisor $\mathcal{D}$ (i.e., the place where the solutions blow up), as a Riemann surface traced on the abelian surface $\widetilde{A}$ (to be constructed in Theorem 12), must be understood as relations connecting the free parameters as they appear firstly in the expansions (31). This means that (32) must be understood as relations connecting $\alpha$ and $\beta$. Let

$$L^{(r)} = \left\{ \begin{array}{l} \text{polynomials} \quad f = f(z, \ldots, z_5) \\ \text{of degree} \le r, \quad \text{such that} \\ f(z(t)) = t^{-1}(z^{(0)} + \ldots), \\ \text{with} \quad z^{(0)} \neq 0 \quad \text{on} \quad \mathcal{D} \\ \text{and with} \quad z(t) \quad \text{as in} \quad (4) \end{array} \right\} \Big/ [F_k = c_k, k = 1, 2, 3],$$

and let $(f_0, f_1, \ldots, f_{N_r})$ be a basis of $L^{(r)}$. We look for r such that: $g(\mathcal{D}^{(r)}) = N_r + 2, \mathcal{D}^{(r)} \subset \mathbb{P}^{N_r}$. We shall show that it is unnecessary to go beyond $r = 4$.

**Theorem 11.** (a) *The spaces $L^{(r)}$, nested according to weighted degree, are generated as follows*

$$L^{(1)} = \{f_0, f_1, f_2\}, \quad L^{(2)} = L^{(1)} \oplus \{f_3, f_4, f_5, f_6\}, \tag{33}$$
$$L^{(3)} = L^{(2)} \oplus \{f_7, f_8, f_9, f_{10}\}, \quad L^{(4)} = L^{(3)} \oplus \{f_{12}, f_{13}, f_{14}, f_{15}\},$$

*where* $f_0 = 1, f_1 = z_1, f_2 = z_2, f_3 = 2z_5 - z_1^2, f_4 = z_3 + 2\varepsilon z_2^2, f_5 = z_4 + \varepsilon z_1 z_2, f_6 = [f_1, f_2]$, $f_7 = f_1(f_1 + 2\varepsilon f_4), f_8 = f_2(f_1 + 2\varepsilon f_4), f_9 = z_4(f_3 + 2\varepsilon f_6), f_{10} = z_5(f_3 + 2\varepsilon f_6), f_{11} = f_5(f_1 + 2\varepsilon f_4)$, $f_{12} = f_1 f_2(f_3 + 2\varepsilon f_6), f_{13} = f_4 f_5 + [f_1, f_4], f_{14} = [f_1, f_3] + 2\varepsilon [f_1, f_6], f_{15} = f_3 - 2z_5 + 4f_4^2$, *with* $[s_j, s_k] = \dot{s}_j s_k - s_j \dot{s}_k$, *the wronskien of* $s_k$ *and* $s_j$.

(b) $L^{(4)}$ *provides an embedding of* $\mathcal{D}^{(4)}$ *into projective space* $\mathbb{P}^{15}$ *and* $\mathcal{D}^{(4)}$ *has genus 17.*

**Proof.** (a) The proof of (a) is straightforward and can be done by inspection of the expansions (31).

(b) It turns out that neither $L^{(1)}$, nor $L^{(2)}$, nor $L^{(3)}$, yields a Riemann surface of the right genus; in fact $g(\mathcal{D}^{(r)}) \neq \dim L^{(r)} + 1, r = 1, 2, 3$. For instance, the embedding into $\mathbb{P}^2$ via $L^{(1)}$ does not separate the sheets, so we proceed to $L^{(2)}$ and the corresponding embedding into $\mathbb{P}^6$ is unacceptable since $g(\mathcal{D}^{(2)}) - 2 > 6$ and $\mathcal{D}^{(2)} \subset \mathbb{P}^6 \neq \mathbb{P}^{g-2}$, which contradicts the fact that $N_r = g(\mathcal{D}^{(2)}) - 2$. So we proceed to $L^{(3)}$ and we consider the corresponding embedding into $\mathbb{P}^{10}$, according to the functions $(f_0, \ldots, f_{10})$. For finite values of $\alpha$ and $\beta$, dividing the vector $(f_0, \ldots, f_{10})$ by $f_2$ and taking the limit $t \to 0$, to yield $[0 : 2\varepsilon\alpha : 1 : -\varepsilon(4\beta - \alpha^3) : -\alpha : -\varepsilon\alpha^2 : \frac{1}{2}(4\beta - \alpha^3) : \varepsilon\alpha^3 : \frac{1}{2}\alpha^2 : \frac{1}{4}\varepsilon\alpha^3(4\beta - \alpha^3) : -\frac{1}{4}\varepsilon\alpha^4(4\beta - \alpha^3)]$. The point $\alpha = 0$ require special attention. Indeed near $\alpha = 0$, the parameter $\beta$ behaves as follows: $\beta \sim 0, i\sqrt{c_3}, -i\sqrt{c_3}$. Thus near $(\alpha, \beta) = (0, 0)$, the corresponding point is mapped into the point $[0 : 0 : 1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0]$ in $\mathbb{P}^{10}$ which is independent of $\varepsilon = \pm 1$, whereas near the point $(\alpha, \beta) = (0, i\sqrt{c_3})$ (resp. $(\alpha, \beta) = (0, -i\sqrt{c_3})$) leads to two different points: $[0 : 0 : 1 : -4\varepsilon i\sqrt{c_3} : 0 : 0 : 2\varepsilon i\sqrt{c_3} : 0 : 0 : 0 : 0]$ (resp. $[0 : 0 : 1 : 4\varepsilon i\sqrt{c_3} : 0 : 0 : -2\varepsilon i\sqrt{c_3} : 0 : 0 : 0 : 0]$), according to the sign of $\varepsilon$. The Riemann surface (31) has three points covering $\alpha = \infty$, at which $\beta$ behaves as follows: $\beta \sim -\frac{1279}{216}\alpha^3, \frac{1}{432}\alpha^3\left(1333 - 1295i\sqrt{3}\right), \frac{1}{432}\alpha^3\left(1333 + 1295i\sqrt{3}\right)$. Then by dividing the vector $(f_0, \ldots, f_{10})$ by $f_{10}$, the corresponding point is mapped into the point $[0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 1]$ in $\mathbb{P}^{10}$. Thus, $g(\mathcal{D}^{(3)}) - 2 > 10$ and $\mathcal{D}^{(2)} \subset \mathbb{P}^{10} \neq \mathbb{P}^{g-2}$, which contradicts the fact that $N_r = g(\mathcal{D}^{(3)}) - 2$. Consider now the embedding $\mathcal{D}^{(4)}$ into $\mathbb{P}^{15}$ using the 16 functions $f_0, \ldots, f_{15}$ of $L^{(4)}$(33). It is easily seen that these functions separate all points of the Riemann surface (except perhaps for the points at $\alpha = \infty$ and $\alpha = \beta = 0$): The Riemann surfaces $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ are disjoint for finite values of $\alpha$ and $\beta$ except for $\alpha = \beta = 0$; dividing the vector $(f_0, \ldots, f_{15})$ by $f_2$ and taking the limit $t \to 0$, to yield $[0 : 2\varepsilon\alpha : 1 : -\varepsilon(4\beta - \alpha^3) : -\alpha : -\varepsilon\alpha^2 : \frac{1}{2}(4\beta - \alpha^3) : \varepsilon\alpha^3 : \frac{1}{2}\alpha^2 : \frac{1}{4}\varepsilon\alpha^3(4\beta - \alpha^3) : -\frac{1}{4}\varepsilon\alpha^4(4\beta - \alpha^3) : -\frac{1}{2}\varepsilon\alpha^4 : -\frac{1}{4}\alpha^3(4\beta - \alpha^3) : \frac{3}{4}\alpha\left(4\beta - \alpha^3\right) : \varepsilon\alpha^3\left(4\beta - \alpha^3\right) : -2\varepsilon\alpha^3]$. As before, the point $\alpha = 0$ requires special attention and the parameter $\beta$ behaves as follows: $\beta \sim 0, i\sqrt{c_3}, -i\sqrt{c_3}$. Thus near $(\alpha, \beta) = (0, 0)$, the corresponding point is mapped into the point $[0 : 0 : 1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0]$ in $\mathbb{P}^{15}$ which is independent of $\varepsilon = \pm 1$, whereas near the point $(\alpha, \beta) = (0, i\sqrt{c_3})$ (resp. $(\alpha, \beta) = (0, -i\sqrt{c_3})$) leads to two different points : $[0 : 0 : 1 : -4\varepsilon i\sqrt{c_3} : 0 : 0 : 2\varepsilon i\sqrt{c_3} : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0]$ (resp. $[0 : 0 : 1 : 4\varepsilon i\sqrt{c_3} : 0 : 0 : -2\varepsilon i\sqrt{c_3} : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0]$), according to the sign of $\varepsilon$. About the point $\alpha = \infty$, it is appropriate to divide by $f_{10}$; then the corresponding point is mapped into the point $[0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 0 : 0]$, in $\mathbb{P}^{15}$ which is independent of $\varepsilon$. The divisor $\mathcal{D}^{(4)}$ obtained in this way has genus 17 and $\mathcal{D}^{(4)} \subset \mathbb{P}^{15} = \mathbb{P}^{g-2}$, as desired. This ends the proof of the theorem.

Let $\mathcal{L} = L^{(4)}$ and $\mathcal{D} = \mathcal{D}^{(4)}$. Next we wish to construct a surface strip around $\mathcal{D}$ which will support the commuting vector fields. In fact, $\mathcal{D}$ has a good chance to be very ample divisor on an abelian surface, still to be constructed.

**Theorem 12.** *The variety* $A(30)$ *generically is the affine part of an abelian surface* $\widetilde{A}$. *The reduced divisor at infinity* $\widetilde{A} \backslash A = \mathcal{C}_1 + \mathcal{C}_{-1}$, *consists of two copies* $\mathcal{C}_1$ *and* $\mathcal{C}_{-1}$ *of the same genus 7 Riemann surface* $\mathcal{C}(32)$. *The system of differential equations (28) is algebraically completely integrable and the corresponding flows evolve on* $\widetilde{A}$.

**Proof.** We need to attaches the affine part of the intersection of the three invariants $F_1, F_2, F_3$ so as to obtain a smooth compact connected surface in $\mathbb{P}^{15}$. To be precise, the orbits of the vector field (28) running through $\mathcal{D}$ form a smooth surface $\Sigma$ near $\mathcal{D}$ such that $\Sigma \backslash A \subseteq \widetilde{A}$ and the variety $\widetilde{A} = A \cup \Sigma$ is smooth, compact and connected. Indeed, let $\psi(t,p) = \{z(t) = (z_1(t), \ldots, z_5(t)) : t \in \mathbb{C}, 0 < |t| < \varepsilon\}$, be the orbit of the vector field (28) going through the point $p \in A$. Let $\Sigma_p \subset \mathbb{P}^{15}$ be the surface element formed by the divisor $\mathcal{D}$ and the orbits going through $p$, and set $\Sigma \equiv \cup_{p\in\mathcal{D}}\Sigma_p$. Consider the Riemann surface $\mathcal{D}' = \mathcal{H} \cap \Sigma$ where $\mathcal{H} \subset \mathbb{P}^{15}$ is a hyperplane transversal to the direction of the flow. If $\mathcal{D}'$ is smooth, then using the implicit function theorem the surface $\Sigma$ is smooth. But if $\mathcal{D}'$ is singular at 0, then $\Sigma$ would be singular along the trajectory ($t-$axis) which go immediately into the affine part A. Hence, A would be singular which is a contradiction because A is the fibre of a morphism from $\mathbb{C}^5$ to $\mathbb{C}^3$ and so smooth for almost all the three constants of the motion $c_k$. Next, let $\overline{A}$ be the projective closure of A into $\mathbb{P}^5$, let $Z = [Z_0 : Z_1 : \ldots : Z_5] \in \mathbb{P}^5$ and let $I = \overline{A} \cap \{Z_0 = 0\}$ be the locus at infinity. Consider the map $\overline{A} \subseteq \mathbb{P}^5 \to \mathbb{P}^{15}$, $Z \mapsto f(Z)$, where $f = (f_0, f_1, \ldots, f_{15}) \in \mathcal{L}(\mathcal{D})$ and let $\widetilde{A} = f(\overline{A})$. In a neighbourhood $V(p) \subseteq \mathbb{P}^{15}$ of $p$, we have $\Sigma_p = \widetilde{A}$ and $\Sigma_p \backslash \mathcal{D} \subseteq A$. Otherwise there would exist an element of surface $\Sigma_p' \subseteq \widetilde{A}$ such that $\Sigma_p \cap \Sigma_p' = (t - axis)$, orbit $\psi(t,p) = (t - axis)\backslash\, p \subseteq A$, and hence A would be singular along the $t-$axis which is impossible. Since the variety $\overline{A} \cap \{Z_0 \neq 0\}$ is irreducible and since the generic hyperplane section $\mathcal{H}_{gen.}$ of $\overline{A}$ is also irreducible, all hyperplane sections are connected and hence I is also connected. Now, consider the graph $\Gamma_f \subseteq \mathbb{P}^5 \times \mathbb{P}^{15}$ of the map $f$, which is irreducible together with $\overline{A}$. It follows from the irreducibility of I that a generic hyperplane section $\Gamma_f \cap \{\mathcal{H}_{gen.} \times \mathbb{P}^{15}\}$ is irreducible, hence the special hyperplane section $\Gamma_f \cap \{\{Z_0 = 0\} \times \mathbb{P}^{15}\}$ is connected and therefore the projection map $proj_{\mathbb{P}^{15}}\{\Gamma_f \cap \{\{Z_0 = 0\} \times \mathbb{P}^{15}\}\} = f(I) \equiv \mathcal{D}$, is connected. Hence, the variety $A \cup \Sigma = \widetilde{A}$ is compact, connected and embeds smoothly into $\mathbb{P}^{15}$ via $f$. We wish to show that $\widetilde{A}$ is an abelian surface equipped with two everywhere independent commuting vector fields. For doing that, let $\phi^{\tau_1}$ and $\phi^{\tau_2}$ be the flows corresponding to vector fields $X_{F_1}$ and $X_{F_2}$. The latter are generated respectively by $F_1$ and $F_2$. For $p \in \mathcal{D}$ and for small $\varepsilon > 0$, $\phi^{\tau_1}(p), \forall\tau_1, 0 < |\tau_1| < \varepsilon$, is well defined and $\phi^{\tau_1}(p) \in \widetilde{A}\backslash A$. Then we may define $\phi^{\tau_2}$ on $\widetilde{A}$ by $\phi^{\tau_2}(q) = \phi^{-\tau_1}\phi^{\tau_2}\phi^{\tau_1}(q), q \in U(p) = \phi^{-\tau_1}(U(\phi^{\tau_1}(p)))$, where $U(p)$ is a neighbourhood of $p$. By commutativity one can see that $\phi^{\tau_2}$ is independent of $\tau_1$; $\phi^{-\tau_1-\varepsilon_1}\phi^{\tau_2}\phi^{\tau_1+\varepsilon_1}(q) = \phi^{-\tau_1}\phi^{-\varepsilon_1}\phi^{\tau_2}\phi^{\tau_1}\phi^{\varepsilon_1} = \phi^{-\tau_1}\phi^{\tau_2}\phi^{\tau_1}(q)$. We affirm that $\phi^{\tau_2}(q)$ is holomorphic away from $\mathcal{D}$. This because $\phi^{\tau_2}\phi^{\tau_1}(q)$ is holomorphic away from $\mathcal{D}$ and that $\phi^{\tau_1}$ is holomorphic in $U(p)$ and maps bi-holomorphically $U(p)$ onto $U(\phi^{\tau_1}(p))$. Now, since the flows $\phi^{\tau_1}$ and $\phi^{\tau_2}$ are holomorphic and independent on $\mathcal{D}$, we can show along the same lines as in the Arnold-Liouville theorem [15] that $\widetilde{A}$ is a complex torus $\mathbb{C}^2/lattice$ and so in particular $\widetilde{A}$ is a Kähler variety. And that will done, by considering the local diffeomorphism $\mathbb{C}^2 \to \widetilde{A}, (\tau_1, \tau_2) \mapsto \phi^{\tau_1}\phi^{\tau_2}(p)$, for a fixed origin $p \in A$. The additive subgroup $\{(\tau_1, \tau_2) \in \mathbb{C}^2 : \phi^{\tau_1}\phi^{\tau_2}(p) = p\}$ is a lattice of $\mathbb{C}^2$, hence $\mathbb{C}^2/lattice \to \widetilde{A}$ is a biholomorphic diffeomorphism and $\widetilde{A}$ is a Kähler variety with Kähler metric given by $d\tau_1 \otimes d\overline{\tau}_1 + d\tau_2 \otimes d\overline{\tau}_2$. As mentioned in appendix A, a compact complex Kähler variety having the required number as (its dimension) of independent meromorphic functions is a projective variety. In fact, here we have $\widetilde{A} \subseteq \mathbb{P}^{15}$. Thus $\widetilde{A}$ is both a projective variety and a complex torus $\mathbb{C}^2/lattice$ and hence an abelian surface as a consequence of Chow theorem. This completes the proof of the theorem.

**Remark 3.1.** (a) *Note that the reflection $\sigma$ on the affine variety $A$ amounts to the flip $\sigma$: $(z_1, z_2, z_3, z_4, z_5) \mapsto (z_1, -z_2, z_3, -z_4, z_5)$, changing the direction of the commuting vector fields. It can be extended to the (-Id)-involution about the origin of $\mathbb{C}^2$ to the time flip $(t_1, t_2) \mapsto (-t_1, -t_2)$ on $\widetilde{A}$, where $t_1$ and $t_2$ are the time coordinates of each of the flows $X_{F_1}$ and $X_{F_2}$. The involution $\sigma$ acts on the parameters of the Laurent solution (30) as follows $\sigma : (t, \alpha, \beta, \gamma, \theta) \longmapsto (-t, -\alpha, -\beta, -\gamma, \theta)$, interchanges the Riemann surfaces $\mathcal{C}_\varepsilon$ and the linear space $\mathcal{L}$ can be split into a direct sum of even and odd functions. Geometrically, this involution interchanges $\mathcal{C}_1$ and $\mathcal{C}_{-1}$, i.e., $\mathcal{C}_{-1} = \sigma\mathcal{C}_1$.*

(b) *Consider on $\widetilde{A}$ the holomorphic 1-forms $dt_1$ and $dt_2$ defined by $dt_i(X_{F_j}) = \delta_{ij}$, where $X_{F_1}$ and $X_{F_2}$ are the vector fields generated respectively by $F_1$ and $F_2$. Taking the differentials of $\zeta = 1/z_1$ and $\xi = z_1/z_2$ viewed as functions of $t_1$ and $t_2$, using the vector fields and the Laurent* *series (31) and solving linearly for $dt_1$ and $dt_2$, we obtain the holomorphic differentials*

$$\omega_1 = dt_1|_{\mathcal{C}_\varepsilon} = \frac{1}{\triangle}\left(\frac{\partial\xi}{\partial t_2}d\zeta - \frac{\partial\zeta}{\partial t_2}d\xi\right)\Big|_{\mathcal{C}_\varepsilon} = \frac{8}{\alpha\left(-4\beta+\alpha^3\right)}d\alpha,$$

$$\omega_2 = dt_2|_{\mathcal{C}_\varepsilon} = \frac{1}{\triangle}\left(\frac{-\partial\xi}{\partial t_1}d\zeta - \frac{\partial\zeta}{\partial t_1}d\xi\right)\Big|_{\mathcal{C}_\varepsilon} = \frac{2}{\left(-4\beta+\alpha^3\right)^2}d\alpha,$$

*with* $\Delta \equiv \frac{\partial\zeta}{\partial t_1}\frac{\partial\xi}{\partial t_2} - \frac{\partial\zeta}{\partial t_2}\frac{\partial\xi}{\partial t_1}$. *The zeroes of $\omega_2$ provide the points of tangency of the vector field $X_{F_1}$ to $\mathcal{C}_\varepsilon$. We have* $\frac{\omega_1}{\omega_2} = \frac{4}{\alpha}\left(-4\beta+\alpha^3\right)$, *and $X_{F_1}$ is tangent to $\mathcal{H}_\varepsilon$ at the point covering $\alpha = \infty$.*

### *3.2. The Hénon-Heiles System*

The Hénon-Heiles system

$$\dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad q_2 = \frac{\partial H}{\partial p_2}, \quad \dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2}, \tag{34}$$

with

$$H \equiv H_1 = \frac{1}{2}\left(p_1^2 + p_2^2 + aq_1^2 + bq_2^2\right) + q_1^2 q_2 + 6q_2^3,$$

has another constant of motion

$$H_2 = q_1^4 + 4q_1^2q_2^2 - 4p_1\left(p_1q_2 - p_2q_1\right) + 4aq_1^2q_2 + (4a-b)\left(p_1^2 + aq_1^2\right),$$

where $a$, $b$, are constant parameters and $q_1, q_2, p_1, p_2$ are canonical coordinates and momenta, respectively. First studied as a mathematical model to describe the chaotic motion of a test star in an axisymmetric galactic mean gravitational field this system is widely explored in other branches of physics. It well-known from applications in stellar dynamics, statistical mechanics and quantum mechanics. It provides a model for the oscillations of atoms in a three-atomic molecule. The system (34) possesses Laurent series solutions depending on 3 free parameters $\alpha, \beta, \gamma$, namely

$$q_1 = \frac{\alpha}{t} + \left(\frac{\alpha^3}{12} + \frac{\alpha A}{2} - \frac{\alpha B}{12}\right)t + \beta t^2 + q_1^{(4)}t^3 + q_1^{(5)}t^4 + q_1^{(6)}t^5 + \cdots,$$

$$q_2 = -\frac{1}{t^2} + \frac{\alpha^2}{12} - \frac{B}{12} + \left(\frac{\alpha^4}{48} + \frac{\alpha^2 A}{10} - \frac{\alpha^2 B}{60} - \frac{B^2}{240}\right)t^2 + \frac{\alpha\beta}{3}t^3 + \gamma t^4 + \cdots,$$

where $p_1 = \dot{q}_1$, $p_2 = \dot{q}_2$ and

$$q_1^{(4)} = \frac{\alpha AB}{24} - \frac{\alpha^5}{72} + \frac{11\alpha^3 B}{720} - \frac{11\alpha^3 A}{120} - \frac{\alpha B^2}{720} - \frac{\alpha A^2}{8}, \quad q_1^{(5)} = -\frac{\beta\alpha^2}{12} + \frac{\beta B}{60} - \frac{A\beta}{10},$$

$$q_1^{(6)} = -\frac{\alpha\gamma}{9} - \frac{\alpha^7}{15552} - \frac{\alpha^5 A}{2160} + \frac{\alpha^5 B}{12960} + \frac{\alpha^3 B^2}{25920} + \frac{\alpha^3 A^2}{1440} - \frac{\alpha^3 AB}{4320} + \frac{\alpha AB^2}{1440} - \frac{\alpha B^3}{19440} - \frac{\alpha A^2 B}{288} + \frac{\alpha A^3}{144}.$$

Let $\mathcal{D}$ be the pole solutions restricted to the surface

$$M_c = \bigcap_{i=1}^{2}\left\{x \equiv (q_1, q_2, p_1, p_2) \in \mathbb{C}^4, H_i(x) = c_i\right\},$$

to be precise $\mathcal{D}$ is the closure of the continuous components of the set of Laurent series solutions $x(t)$ such that $H_i(x(t)) = c_i$, $1 \le i \le 2$, i.e., $\mathcal{D} = t^0$—coefficient of $M_c$. Thus we find an algebraic curve defined by

$$\mathcal{D}: \quad \beta^2 = P_8(\alpha), \tag{35}$$

where

$$P_8\left(\alpha\right)=-\frac{7}{15\,552}\alpha^8-\frac{1}{432}\left(5A-\frac{13}{18}B\right)\alpha^6-\frac{1}{36}\left(\frac{671}{15\,120}B^2+\frac{17}{7}A^2-\frac{943}{1260}BA\right)\alpha^4$$
$$-\frac{1}{36}\left(4A^3-\frac{1}{2520}B^3-\frac{13}{6}A^2B+\frac{2}{9}AB^2-\frac{10}{7}c_1\right)\alpha^2+\frac{1}{36}c_2.$$

The curve $\mathcal{D}$ determined by an eight-order equation is smooth, hyperelliptic and its genus is 3. Moreover, the map

$$\sigma:\quad \mathcal{D}\longrightarrow\mathcal{D},\quad (\beta,\alpha)\longmapsto(\beta,-\alpha), \tag{36}$$

is an involution on $\mathcal{D}$ and the quotient $\mathcal{E}=\mathcal{D}/\sigma$ is an elliptic curve defined by

$$\mathcal{E}:\quad \beta^2=P_4(\zeta), \tag{37}$$

where $P_4\left(\zeta\right)$ is the degree 4 polynomial in $\zeta=\alpha^2$ obtained from (35). The hyperelliptic curve $\mathcal{D}$ is thus a 2-sheeted ramified covering of the elliptic curve $\mathcal{E}$ (37),

$$\rho:\quad \mathcal{D}\longrightarrow\mathcal{E},\quad (\beta,\alpha)\longmapsto(\beta,\zeta), \tag{38}$$

ramified at the four points covering $\zeta=0$ and $\infty$. The affine surface $M_c$ completes into an abelian surface $\widetilde{M_c}$, by adjoining the divisor $\mathcal{D}$. The latter defines on $\widetilde{M_c}$ a polarization $(1,2)$. The divisor $2\mathcal{D}$ is very ample and the functions $1, y_1, y_1^2, y_2, x_1, x_1^2+y_1^2y_2, x_2y_1-2x_1y_2, x_1x_2+2Ay_1y_2+2y_1y_2^2$, embed $\widetilde{M_c}$ smoothly into $\mathbb{CP}^7$ with polarization $(2,4)$. Then the system (34) is algebraically completely integrable and the corresponding flow evolues on an abelian surface $\widetilde{M_c}=\mathbb{C}^2/lattice$, where the lattice is generated by the period matrix $\begin{pmatrix}2&0&a&c\\0&4&c&b\end{pmatrix}$, $\operatorname{Im}\begin{pmatrix}a&c\\c&b\end{pmatrix}>0$.

**Theorem 13.** *The abelian surface $\widetilde{M_c}$ which completes the affine surface $M_c$ is the dual Prym variety $Prym^*\left(\mathcal{D}/\mathcal{E}\right)$ of the genus 3 hyperelliptic curve $\mathcal{D}$ (35) for the involution $\sigma$ interchanging the sheets of the double covering $\rho$ (38) and the problem linearizes on this variety.*

**Proof.** Let $(a_1,a_2,a_3,b_1,b_2,b_3)$ be a canonical homology basis of $\mathcal{D}$ such that $\sigma\left(a_1\right)=a_3$, $\sigma\left(b_1\right)=b_3$, $\sigma\left(a_2\right)=-a_2$, $\sigma\left(b_2\right)=-b_2$, for the involution $\sigma$ (36). As a basis of holomorphic differentials $\omega_0,\omega_1,\omega_2$ on the curve $\mathcal{D}$ (35) we take the differentials $\omega_1=\dfrac{\alpha^2d\alpha}{\beta},\omega_2=\dfrac{d\alpha}{\beta},\omega_3=\dfrac{\alpha d\alpha}{\beta}$, and obviously $\sigma^*(\omega_1)=-\omega_1,\sigma^*(\omega_2)=-\omega_2,\sigma^*(\omega_3)=\omega_3$. Recall that the Prym variety $Prym\left(\mathcal{D}/\mathcal{E}\right)$ is a subabelian variety of the Jacobi variety $Jac(\mathcal{D})=Pic^0(\mathcal{D})=H^1(\mathcal{O}_{\mathcal{D}})\ /\ H^1(\mathcal{D},\mathbb{Z})$ constructed from the double cover $\rho$: the involution $\sigma$ on $\mathcal{D}$ interchanging sheets, extends by linearity to a map $\sigma$: $Jac(\mathcal{D})\to Jac(\mathcal{D})$ and up to some points of order two, $Jac(\mathcal{D})$ splits into an even part and an odd part: the even part is an elliptic curve (the quotient of $\mathcal{D}$ by $\sigma$, i.e., $\mathcal{E}$ (18)) and the odd part is a 2-dimensional abelian surface $Prym\left(\mathcal{D}/\mathcal{E}\right)$. We consider the period matrix $\Omega$ of $Jac(D)$

$$\Omega=\begin{pmatrix}\int_{a_1}\omega_1&\int_{a_2}\omega_1&\int_{a_3}\omega_1&\int_{b_1}\omega_1&\int_{b_2}\omega_1&\int_{b_3}\omega_1\\ \int_{a_1}\omega_2&\int_{a_2}\omega_2&\int_{a_3}\omega_2&\int_{b_1}\omega_2&\int_{b_2}\omega_2&\int_{b_3}\omega_2\\ \int_{a_1}\omega_3&\int_{a_2}\omega_3&\int_{a_3}\omega_3&\int_{b_1}\omega_3&\int_{b_2}\omega_3&\int_{b_3}\omega_3\end{pmatrix}.$$

Then,

$$\Omega=\begin{pmatrix}\int_{a_1}\omega_1&\int_{a_2}\omega_1&-\int_{a_1}\omega_1&\int_{b_1}\omega_1&\int_{b_2}\omega_1&-\int_{b_1}\omega_1\\ \int_{a_1}\omega_2&\int_{a_2}\omega_2&-\int_{a_1}\omega_2&\int_{b_1}\omega_2&\int_{b_2}\omega_2&-\int_{b_1}\omega_2\\ \int_{a_1}\omega_3&0&\int_{a_1}\omega_3&\int_{b_1}\omega_3&0&\int_{b_1}\omega_3\end{pmatrix},$$

and therefore the period matrices of $Jac(\mathcal{E})$(i.e., $\mathcal{E}$), $Prym(\mathcal{D}/\mathcal{E})$ and $Prym^*(\mathcal{D}/\mathcal{E})$ are respectively $\Delta = (\int_{a_1} \omega_3 \ \int_{b_1} \omega_3)$,

$$\Gamma = \begin{pmatrix} 2\int\limits_{a_1} \omega_1 & \int\limits_{a_2} \omega_1 & 2\int\limits_{b_1} \omega_1 & \int\limits_{b_2} \omega_1 \\ 2\int\limits_{a_1} \omega_2 & \int\limits_{a_2} \omega_2 & 2\int\limits_{b_1} \omega_2 & \int\limits_{b_2} \omega_2 \end{pmatrix},$$

and

$$\Gamma^* = \begin{pmatrix} \int\limits_{a_1} \omega_1 & \int\limits_{a_2} \omega_1 & \int\limits_{b_1} \omega_1 & \int\limits_{b_2} \omega_1 \\ \int\limits_{a_1} \omega_2 & \int\limits_{a_2} \omega_2 & \int\limits_{b_1} \omega_2 & \int\limits_{b_2} \omega_2 \end{pmatrix}.$$

Let $L_\Omega = \left\{ \sum_{i=1}^3 m_i \int_{a_i} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + n_i \int_{b_i} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} : m_i, n_i \in \mathbb{Z} \right\}$, be the period lattice associated to $\Omega$. Let us denote also by $L_\Delta$, the period lattice associated $\Delta$. We have the following diagram

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{E} & & \mathcal{D} & & \\ & & & & \downarrow \varphi^* & \swarrow & \downarrow \varphi & & \\ 0 & \longrightarrow & \ker N_\varphi & \longrightarrow & Prym(\mathcal{D}/\mathcal{E} \oplus \mathcal{E} = Jac(\mathcal{D}) & \overset{N_\varphi}{\longrightarrow} & \mathcal{E} & \longrightarrow & 0 \\ & & & \searrow \tau & \downarrow & & & & \\ & & & & \widetilde{M}_c = M_c \cup 2\mathcal{D} \simeq \mathbb{C}^2/lattice & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

The polarization map $\tau$: $Prym(\mathcal{D}/\mathcal{E}) \longrightarrow \widetilde{M}_c = Prym^*(\mathcal{D}/\mathcal{E})$, has kernel $(\varphi^*\mathcal{E}) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and the induced polarization on $Prym(\mathcal{D}/\mathcal{E})$ is of type $(1, 2)$. Let $\widetilde{M}_c \to \mathbb{C}^2/L_\Lambda : p \curvearrowright \int_{p_0}^p \binom{dt_1}{dt_2}$, be the uniformizing map where $dt_1$, $dt_2$ are two differentials on $\widetilde{M}_c$ corresponding to the flows generated respectively by $H_1$, $H_2$ such that: $dt_1|_\mathcal{D} = \omega_1$ and $dt_2|_\mathcal{D} = \omega_2$,

$$L_\Lambda = \left\{ \sum_{k=1}^4 n_k \begin{pmatrix} \int\limits_{\nu_k} dt_1 \\ \int\limits_{\nu_k} dt_2 \end{pmatrix} : n_k \in \mathbb{Z} \right\},$$

is the lattice associated to the period matrix

$$\Lambda = \begin{pmatrix} \int\limits_{\nu_1} dt_1 & \int\limits_{\nu_2} dt_1 & \int\limits_{\nu_4} dt_1 & \int\limits_{\nu_4} dt_1 \\ \int\limits_{\nu_1} dt_2 & \int\limits_{\nu_2} dt_2 & \int\limits_{\nu_3} dt_2 & \int\limits_{\nu_4} dt_2 \end{pmatrix},$$

and $(\nu_1, \nu_2, \nu_3, \nu_4)$ is a basis of $H_1(\widetilde{M}_c, \mathbb{Z})$. By the Lefschetz theorem on hyperplane section [9], the map $H_1(\mathcal{D}, \mathbb{Z}) \longrightarrow H_1(\widetilde{M}_c, \mathbb{Z})$ induced by the inclusion $\mathcal{D} \hookrightarrow \widetilde{M}_c$ is surjective and consequently we can find

4 cycles $\nu_1, \nu_2, \nu_3, \nu_4$ on the curve $\mathcal{D}$ such that

$$\Lambda = \begin{pmatrix} \int\limits_{\nu_1} \omega_1 & \int\limits_{\nu_2} \omega_1 & \int\limits_{\nu_4} \omega_1 & \int\limits_{\nu_4} \omega_1 \\ \int\limits_{\nu_1} \omega_2 & \int\limits_{\nu_2} \omega_2 & \int\limits_{\nu_3} \omega_2 & \int\limits_{\nu_4} \omega_2 \end{pmatrix},$$

and $L_\Lambda = \left\{ \sum_{k=1}^{4} n_k \begin{pmatrix} \int_{\nu_k} \omega_1 \\ \int_{\nu_k} \omega_2 \end{pmatrix} : n_k \in \mathbb{Z} \right\}$. The cycles $\nu_1, \nu_2, \nu_3, \nu_4$ in $\mathcal{D}$ which we look for are $a_1, b_1, a_2, b_2$ and they generate $H_1(\widetilde{M}_c, \mathbb{Z})$ such that

$$\Lambda = \begin{pmatrix} \int\limits_{a_1} \omega_1 & \int\limits_{b_1} \omega_1 & \int\limits_{a_2} \omega_1 & \int\limits_{b_2} \omega_1 \\ \int\limits_{a_1} \omega_2 & \int\limits_{b_1} \omega_2 & \int\limits_{a_2} \omega_2 & \int\limits_{b_2} \omega_2 \end{pmatrix},$$

is a Riemann matrix. We show that $\Lambda = \Gamma^*$, i.e., the period matrix of $Prym^*(\mathcal{D}/\mathcal{E})$ dual of $Prym(\mathcal{D}/\mathcal{E})$. Consequently $\widetilde{M}_c$ and $Prym^*(\mathcal{D}/\mathcal{E})$ are two abelian varieties analytically isomorphic to the same complex torus $\mathbb{C}^2/L_\Lambda$. By Chow's theorem [9], $\widetilde{\mathcal{A}}_c$ and $Prym^*(\mathcal{D}/\mathcal{E})$ are then algebraically isomorphic.

### *3.3. The Kowalewski Rigid Body Motion*

The motion for the Kowalewski's top is governed by the equations

$$\dot{m} = m \wedge \lambda m + \gamma \wedge l, \qquad \dot{\gamma} = \gamma \wedge \lambda m, \tag{39}$$

where $m, \gamma$ and $l$ denote respectively the angular momentum, the directional cosine of the $z$-axis (fixed in space), the center of gravity which after some rescaling and normalization may be taken as $l = (1, 0, 0)$ and $\lambda m = (m_1/2, m_2/2, m_3/2)$. The system (39) can be written

$$\begin{aligned} \dot{m}_1 &= m_2 \quad m_3, & \dot{\gamma}_1 &= 2m_3\gamma_2 - m_2\gamma_3, \\ \dot{m}_2 &= -m_1 \quad m_3 + 2\gamma_3, & \dot{\gamma}_2 &= \quad m_1\gamma_3 - 2m_3\gamma_1, \\ \dot{m}_3 &= -2\gamma_2, & \dot{\gamma}_3 &= m_2\gamma_1 - m_1\gamma_2, \end{aligned} \tag{40}$$

with constants of motion

$$\begin{gathered} H_1 = \frac{1}{2}\left(m_1^2 + m_2^2\right) + m_3^2 + 2\gamma_1 = c_1, \\ H_2 = m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3 = c_2, \quad H_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = c_3 = 1, \\ H_4 = \left(\left(\frac{m_1 + im_2}{2}\right)^2 - (\gamma_1 + i\gamma_2)\right)\left(\left(\frac{m_1 - im_2}{2}\right)^2 - (\gamma_1 - i\gamma_2)\right) = c_4. \end{gathered} \tag{41}$$

The system (40) admits two distinct families of Laurent series solutions:

$$m_1(t) = \begin{cases} \dfrac{\alpha_1}{t} + i\left(\alpha_1^2 - 2\right)\alpha_2 + \circ(t), \\ \dfrac{\alpha_1}{t} - i\left(\alpha_1^2 - 2\right)\alpha_2 + \circ(t), \end{cases} \qquad \gamma_1(t) = \begin{cases} \dfrac{1}{2t^2} + \circ(t), \\ \dfrac{1}{2t^2} + \circ(t), \end{cases}$$

$$m_2(t) = \begin{cases} \dfrac{i\alpha_1}{t} - \alpha_1^2\alpha_2 + \circ(t), \\ \dfrac{-i\alpha_1}{t} - \alpha_1^2\alpha_2 + \circ(t), \end{cases} \qquad \gamma_2(t) = \begin{cases} \dfrac{i}{2t^2} + \circ(t), \\ \dfrac{-i}{2t^2} + \circ(t), \end{cases}$$

$$m_3(t)=\begin{cases}\dfrac{i}{t}+\alpha_1\alpha_2+\circ(t)\,,\\ \dfrac{-i}{t}+\alpha_1\alpha_2+\circ(t)\,,\end{cases}\qquad \gamma_3(t)=\begin{cases}\dfrac{\alpha_2}{t}+\circ(t)\,,\\ \dfrac{\alpha_2}{t}+\circ(t)\,,\end{cases}$$

which depend on 5 free parameters $\alpha_1,\ldots,\alpha_5$. By substituting these series in the constants of the motion $H_i$ (41), one eliminates three parameters linearly, leading to algebraic relation between the two remaining parameters, which is nothing but the equation of the divisor $\mathcal{D}$ along which the $m_i,\gamma_i$ blow up. Since the system (40) admits two families of Laurent solutions, then $\mathcal{D}$ is a set of two isomorphic curves of genus 3, $\mathcal{D}=\mathcal{D}_1+\mathcal{D}_{-1}$:

$$\mathcal{D}_\varepsilon:\quad P(\alpha_1,\alpha_2)=\left(\alpha_1^2-1\right)\left(\left(\alpha_1^2-1\right)\alpha_2^2-P(\alpha_2)\right)+c_4=0, \tag{42}$$

where $P(\alpha_2)=c_1\alpha_2^2-2\varepsilon c_2\alpha_2-1$ and $\varepsilon=\pm1$. Each of the curve $\mathcal{D}_\varepsilon$ is a $2-1$ ramified cover $(\alpha_1,\alpha_2,\beta)$ of elliptic curves $\mathcal{D}_\varepsilon^0$:

$$\mathcal{D}_\varepsilon^0:\quad \beta^2=P^2(\alpha_2)-4c_4\alpha_2^4, \tag{43}$$

ramified at the 4 points $\alpha_1=0$ covering the 4 roots of $P(\alpha_2)=0$. It was shown [12] that each divisor $\mathcal{D}_\varepsilon$ is ample and defines a polarization $(1,2)$, whereas the divisor $\mathcal{D}$, of geometric genus 9, is very ample and defines a polarization $(2,4)$. The affine surface $M_c=\bigcap_{i=1}^4\{H_i=c_i\}\subset\mathbb{C}^6$, defined by putting the four invariants (41) of the Kowalewski flow (40) equal to generic constants, is the affine part of an abelian surface $\widetilde{M_c}$ with

$\widetilde{M_c}\backslash M_c=\mathcal{D}=$ one genus 9 curve consisting of two genus 3
curves $\mathcal{D}_\varepsilon$ (42) intersecting in 4 points. Each
$\mathcal{D}_\varepsilon$ is a double cover of an elliptic curve $\mathcal{D}_\varepsilon^0$ (43)
ramified at 4 points.

Moreover, the Hamiltonian flows generated by the vector fields $X_{H_1}$ and $X_{H_4}$ are straight lines on $\widetilde{M_c}$. The 8 functions 1, $f_1=m_1$, $f_2=m_2$, $f_3=m_3$, $f_4=\gamma_3$, $f_5=f_1^2+f_2^2$, $f_6=4f_1f_4-f_3f_5$, $f_7=(f_2\gamma_1-f_1\gamma_2)f_3+2f_4\gamma_2$, form a basis of the vector space of meromorphic functions on $\widetilde{M_c}$ with at worst a simple pole along $\mathcal{D}$ Moreover, the map

$$\widetilde{M_c}\simeq\mathbb{C}^2/Lattice\to\mathbb{CP}^7,\quad (t_1,t_2)\mapsto[(1,f_1(t_1,t_2),\ldots,f_7(t_1,t_2))]\,,$$

is an embedding of $\widetilde{M_c}$ into $\mathbb{CP}^7$. Following the method (Theorem 13), we obtain the following theorem:

**Theorem 14.** *The tori $\widetilde{M_c}$ can be identified as $\widetilde{M_c}=Prym^*(\mathcal{D}_\varepsilon/\mathcal{D}_\varepsilon^0)$, i.e., dual of $Prym(\mathcal{D}_\varepsilon/\mathcal{D}_\varepsilon^0)$ and the problem linearizes on this Prym variety.*

### *3.4. Kirchhoff's Equations of Motion of a Solid in an Ideal Fluid*

The Kirchhoff's equations of motion of a solid in an ideal fluid have the form

$$\begin{aligned}
\dot p_1&=p_2\frac{\partial H}{\partial l_3}-p_3\frac{\partial H}{\partial l_2}, & \dot l_1&=p_2\frac{\partial H}{\partial p_3}-p_3\frac{\partial H}{\partial p_2}+l_2\frac{\partial H}{\partial l_3}-l_3\frac{\partial H}{\partial l_2},\\
\dot p_2&=p_3\frac{\partial H}{\partial l_1}-p_1\frac{\partial H}{\partial l_3}, & \dot l_2&=p_3\frac{\partial H}{\partial p_1}-p_1\frac{\partial H}{\partial p_3}+l_3\frac{\partial H}{\partial l_1}-l_1\frac{\partial H}{\partial l_3},\\
\dot p_3&=p_1\frac{\partial H}{\partial l_2}-p_2\frac{\partial H}{\partial l_1}, & \dot l_3&=p_1\frac{\partial H}{\partial p_2}-p_2\frac{\partial H}{\partial p_1}+l_1\frac{\partial H}{\partial l_2}-l_2\frac{\partial H}{\partial l_1},
\end{aligned} \tag{44}$$

where $(p_1,p_2,p_3)$ is the velocity of a point fixed relatively to the solid, $(l_1,l_2,l_3)$ the angular velocity of the body expressed with regard to a frame of reference also fixed relatively to the solid and $H$ is the hamiltonian. These equations can be regarded as the equations of the geodesics of the right-invariant metric on the group $E(3)=SO(3)\times\mathbb{R}^3$ of motions of 3-dimensional euclidean space $\mathbb{R}^3$, generated by rotations and translations. Hence the motion has the trivial coadjoint orbit invariants $\langle p,p\rangle$ and $\langle p,l\rangle$. As it turns out, this is a special case of a more general system of equations written as

$$\dot{x} = x\wedge\frac{\partial H}{\partial x} + y\wedge\frac{\partial H}{\partial y},\quad \dot{y} = y\wedge\frac{\partial H}{\partial x} + x\wedge\frac{\partial H}{\partial y},$$

where $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ et $y=(y_1,y_2,y_3)\in\mathbb{R}^3$. The first set can be obtained from the second by putting $(x,y)=(l,p/\varepsilon)$ and letting $\varepsilon\to 0$. The latter set of equations is the geodesic flow on $SO(4)$ for a left invariant metric defined by the quadratic form $H$. In Clebsch's case, equations (44) have the four invariants:

$$H_1 = H = \frac{1}{2}\left(a_1p_1^2+a_2p_2^2+a_3p_3^2+b_1l_1^2+b_2l_2^2+b_3l_3^2\right),\quad H_2 = p_1^2+p_2^2+p_3^2,$$

$$H_3 = p_1l_1+p_2l_2+p_3l_3,\quad H_4 = \frac{1}{2}\left(b_1p_1^2+b_2p_2^2+b_3p_3^2+\varrho\left(l_1^2+l_2^2+l_3^2\right)\right),$$

with $\frac{a_2-a_3}{b_1}+\frac{a_3-a_1}{b_2}+\frac{a_1-a_2}{b_3}=0$, and the constant $\varrho$ satisfies the conditions $\varrho=\frac{b_1(b_2-b_3)}{a_2-a_3}=\frac{b_2(b_3-b_1)}{a_3-a_1}=\frac{b_3(b_1-b_2)}{a_1-a_2}$. The system (44) can be written in the form (11) with $m=6$; to be precise

$$\dot{x} = f(x)\equiv J\frac{\partial H}{\partial x},\quad x=(p_1,p_2,p_3,l_1,l_2,l_3)^{\intercal},\tag{45}$$

where

$$J=\begin{pmatrix} O & P\\ P & L\end{pmatrix}, P=\begin{pmatrix} 0 & -p_3 & p_2\\ p_3 & 0 & -p_1\\ -p_2 & p_1 & 0\end{pmatrix}, L=\begin{pmatrix} 0 & -l_3 & l_2\\ l_3 & 0 & -l_1\\ -l_2 & p_1 & 0\end{pmatrix}.$$

Consider points at infinity which are limit points of trajectories of the flow. In fact, there is a Laurent decomposition of such asymptotic solutions,

$$x(t) = t^{-1}\left(x^{(0)}+x^{(1)}t+x^{(2)}t^2+\dots\right),\tag{46}$$

which depend on $\dim(\textit{phase space})-1=5$ free parameters. Putting (46) into (45), solving inductively for the $x^{(k)}$, one finds at the 0th step a non-linear equation, $x^{(0)}+f(x^{(0)})=0$, and at the $k$th step, a linear system of equations,

$$(L-kI)x^{(k)}=\begin{cases} 0 \quad \text{for} \quad k=1\\ \text{quadratic polynomial in } x^{(1)},\dots,x^{(k)} \quad \text{for} \quad k\geq 1,\end{cases}$$

where $L$ denotes the jacobian map of the non-linear equation above. One parameter appear at the 0th step, i.e., in the resolution of the non-linear equation and the 4 remaining ones at the $k$th step, $k=1,\dots,4$. Taking into account only solutions trajectories lying on the invariant surface $M_c=\bigcap_{i=1}^{4}\{H_i(x)=c_i\}\subset\mathbb{C}^6$, we obtain one-parameter families which are parameterized by a curve:

$$\mathcal{D}:\quad \theta^2+c_1\beta^2\gamma^2+c_2\alpha^2\gamma^2+c_3\alpha^2\beta^2+c_4\alpha\beta\gamma=0,\tag{47}$$

where $\theta$ is an arbitrary parameter and where $\alpha=x_4^{(0)},\beta=x_5^{(0)},\gamma=x_6^{(0)}$ parameterizes the elliptic curve

$$\mathcal{E}:\quad \beta^2=d_1^2\alpha^2-1,\quad \gamma^2=d_2^2\alpha^2+1,\tag{48}$$

with $d_1,d_2$ such that: $d_1^2+d_2^2+1=0$. The curve $\mathcal{D}$ is a 2-sheeted ramified covering of the elliptic curve $\mathcal{E}$. The branch points are defined by the 16 zeroes of $c_1\beta^2\gamma^2+c_2\alpha^2\gamma^2+c_3\alpha^2\beta^2+c_4\alpha\beta\gamma$ on $\mathcal{E}$. The curve $\mathcal{D}$ is unramified at infinity and by Hurwitz's formula, the genus of $\mathcal{D}$ is 9. Upon putting $\zeta\equiv\alpha^2$, the curve $\mathcal{D}$ can also be seen as a 4-sheeted unramified covering of the following curve of genus 3:

$$C:\left(\theta^2+c_1\beta^2\gamma^2+\left(c_2\gamma^2+c_3\beta^2\right)\zeta\right)^2-c_4^2\zeta\beta^2\gamma^2=0.$$

Moreover, the map $\tau\colon C \to C$, $(\theta,\zeta) \mapsto (-\theta,\zeta)$, is an involution on $C$ and the quotient $C_0 = C/\tau$ is an elliptic curve defined by

$$C_0: \quad \eta^2 = c_4^2\zeta\left(d_1^2d_2^2\zeta^2 + \left(d_1^2 - d_2^2\right)\zeta - 1\right).$$

The curve $C$ is a double ramified covering of $C_0$, $C \to C_0$, $(\theta,\eta,\zeta) \mapsto (\eta,\zeta)$,

$$C: \begin{cases} \theta^2 = -c_1\beta^2\gamma^2 - \left(c_2\gamma^2 + c_3\beta^2\right)\zeta + \eta \\ \eta^2 = c_4^2\zeta\left(d_1^2d_2^2\zeta^2 + \left(d_1^2 - d_2^2\right)\zeta - 1\right). \end{cases}$$

Let $(a_1,a_2,a_3,b_1,b_2,b_3)$ be a canonical homology basis of $C$ such that $\tau(a_1) = a_3$, $\tau(b_1) = b_3$, $\tau(a_2) = -a_2$ and $\tau(b_2) = -b_2$ for the involution $\tau$. Using the Poincaré residu map, we show that

$$\omega_0 = \frac{d\zeta}{\eta}, \quad \omega_1 = \frac{\zeta d\zeta}{\theta\eta}, \quad \omega_2 = \frac{d\zeta}{\theta\eta},$$

form a basis of holomorphic differentials on $C$ and $\tau^*(\omega_0) = \omega_0, \tau^*(\omega_k) = -\omega_k$ $(k = 1,2)$. The flow evolues on an abelian surface $\widetilde{M_c} \subseteq \mathbb{CP}^7$ of period matrix $\begin{pmatrix} 2 & 0 & a & c \\ 0 & 4 & c & b \end{pmatrix}$, $\operatorname{Im}\begin{pmatrix} a & c \\ c & b \end{pmatrix} > 0$. Following the method (Theorem 13), we obtain

**Theorem 15.** *The abelian surface $\widetilde{M_c}$ can be identified as $Prym(C/C_0)$. More precisely*

$$\bigcap_{i=1}^{4}\left\{x \in \mathbb{C}^6, H_i(x) = c_i\right\} = Prym(C/C_0)\backslash\mathcal{D},$$

*where $\mathcal{D}$ is a genus 9 curve (47), which is a ramified cover of an elliptic curve $\mathcal{E}$ (48) with 16 branch points.*

## 4. GENERALIZED ALGEBRAIC COMPLETELY INTEGRABLE SYSTEMS

Some others integrable systems appear as coverings of algebraic completely integrable systems. The manifolds invariant by the complex flows are coverings of abelian varieties and these systems are called algebraic completely integrable in the generalized sense.

Consider the case $F_3 = 0$, (see Section 3.1) and the following change of variables

$$z_1 = q_1^2, \quad z_2 = q_2, \quad z_3 = p_2, \quad z_4 = p_1q_1, \quad z_5 = p_1^2 - q_1^2q_2^2.$$

Substituting this into the constants of motion $F_1, F_2, F_3$ leads obviously to the relations

$$H_1 = \frac{1}{2}p_1^2 - \frac{3}{2}q_1^2q_2^2 + \frac{1}{2}p_2^2 - \frac{1}{4}q_1^4 - 2q_2^4, \tag{49}$$

$$H_2 = p_1^4 - 6q_1^2q_2^2p_1^2 + q_1^4q_2^4 - q_1^4p_1^2 + q_1^6q_2^2 + 4q_1^3q_2p_1p_2 - q_1^4p_2^2 + \frac{1}{4}q_1^8,$$

whereas the last constant leads to an identity. Using the differential equations (28) combined with the transformation above leads to the system of differential equations

$$\ddot{q}_1 = q_1\left(q_1^2 + 3q_2^2\right), \quad \ddot{q}_1 = q_2\left(3q_1^2 + 8q_2^2\right). \tag{50}$$

The last equation (28) for $z_5$ leads to an identity. Thus, we obtain the potential constructed by Ramani, Dorozzi and Grammaticos [7, 25]. Evidently, the functions $H_1$ and $H_2$ commute: $\{H_1, H_2\} = 0$. The system (50) is weight-homogeneous with $q_1, q_2$ having weight 1 and $p_1, p_2$ weight 2, so that $H_1$ and $H_2$ have weight 4 and 8 respectively. When one examines all possible singularities, one finds that it possible for the variable $q_1$ to contain square root terms of the type $t^{1/2}$, which are strictly not allowed by the Painlevé test (i.e., the general solutions have no movable singularities other than poles). However, these terms are trivially removed by introducing the variables $z_1, \ldots, z_5$ which restores the Painlevé property to the system. Let B be the affine variety defined by

$$B = \bigcap_{k=1}^{2} \{z \in \mathbb{C}^4 : H_k(z) = b_k\}, \tag{51}$$

where $(b_1, b_2) \in \mathbb{C}^2$.

**Theorem 16.** (a) *The system (50) admits Laurent solutions in $t^{1/2}$, depending on 3 free parameters: $u, v$ and $w$. These solutions restricted to the surface $B(51)$ are parameterized by two copies $\Gamma_1$ and $\Gamma_{-1}$ of the same Riemann surface of genus 16.*

(b) *The invariant surface $B$ (51) can be completed as a cyclic double cover $\overline{B}$ of the abelian surface $\widetilde{A}$, ramified along the divisor $\mathcal{C}_1 + \mathcal{C}_{-1}$. The system (50) is algebraic complete integrable in the generalized sense. Moreover, $\overline{B}$ is smooth except at the point lying over the singularity (of type $A_3$) of $\mathcal{C}_1 + \mathcal{C}_{-1}$ and the resolution $\widetilde{B}$ of $\overline{B}$ is a surface of general type with invariants: $\mathcal{X}(\widetilde{B}) = 1$ and $p_g(\widetilde{B}) = 2$.*

**Proof.** (a) The system (50) possesses 3-dimensional family of Laurent solutions (principal balances) depending on three free parameters $u, v$ and $w$. There are precisely two such families, labeled by $\varepsilon = \pm 1$, and they are explicitly given as follows

$$q_1 = \frac{1}{\sqrt{t}}\left(u - \frac{1}{4}u^3 t + vt^2 - \frac{5}{128}u^7 t^3 + \frac{1}{8}u\left(\frac{3}{4}u^3 v - \frac{7}{256}u^8 + 3\varepsilon w\right)t^4 + \cdots\right), \tag{52}$$

$$q_2 = \frac{1}{t}\left(\frac{1}{2}\varepsilon - \frac{1}{4}\varepsilon u^2 t + \frac{1}{8}\varepsilon u^4 t^2 + \frac{1}{4}\varepsilon u\left(\frac{1}{32}u^5 - 3v\right)t^3 + wt^4 + \cdots\right),$$

$$p_1 = \frac{1}{2t\sqrt{t}}\left(-u - \frac{1}{4}u^3 t + 3vt^2 - \frac{25}{128}t^3 u^7 + \frac{7}{8}u\left(\frac{3}{4}u^3 v - \frac{7}{256}u^8 + 3\varepsilon w\right)t^4 + \cdots\right),$$

$$p_2 = \frac{1}{t^2}\left(-\frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon u^4 t^2 + \frac{1}{2}\varepsilon u\left(\frac{1}{32}u^5 - 3v\right)t^3 + 3wt^4 + \cdots\right).$$

These formal series solutions are convergent as a consequence of the majorant method. By substituting these series in the constants of the motion $H_1 = b_1$ and $H_2 = b_2$, one eliminates the parameter $w$ linearly, leading to an equation connecting the two remaining parameters $u$ and $v$:

$$\begin{aligned}\Gamma: \quad & \frac{65}{4}uv^3 + \frac{93}{64}u^6 v^2 + \frac{3}{8192}\left(-9829u^8 + 26\,112H_1\right)u^3 v \\ & - \frac{10\,299}{65\,536}u^{16} - \frac{123}{256}H_1 u^8 + H_2 + \frac{1\,536\,298\,731}{52} = 0.\end{aligned} \tag{53}$$

According to Hurwitz' formula, this defines a Riemann surface $\Gamma$ of genus 16. The Laurent solutions restricted to the surface $B(51)$ are thus parameterized by two copies $\Gamma_{-1}$ and $\Gamma_1$ of the same Riemann surface $\Gamma$.

(b) The morphism $\varphi$: $B \longrightarrow A$, $(q_1, q_2, p_1, p_2) \longmapsto (z_1, z_2, z_3, z_4, z_5)$, maps the vector field (50) into an algebraic completely integrable system (1) in five unknowns and the affine variety $B(51)$ onto the affine part $A(30)$ of an abelian variety $\widetilde{A}$ with $\widetilde{A}\backslash A = \mathcal{C}_1 + \mathcal{C}_{-1}$. Observe that $\varphi$ is an unramified cover. The Riemann surface $\Gamma(53)$ play an important role in the construction of a compactification $\overline{B}$ of $B$. Let us denote by $G$ a cyclic group of two elements $\{-1, 1\}$ on $V_\varepsilon^j = U_\varepsilon^j \times \{\tau \in \mathbb{C} : 0 < |\tau| < \delta\}$, where $\tau = t^{1/2}$ and $U_\varepsilon^j$ is an affine chart of $\Gamma_\varepsilon$ for which the Laurent solutions (52) are defined. The action of $G$ is defined by $(-1) \circ (u, v, \tau) = (-u, -v, -\tau)$ and is without fixed points in $V_\varepsilon^j$. So we can identify the quotient $V_\varepsilon^j/G$ with the image of the smooth map $h_\varepsilon^j$: $V_\varepsilon^j \to B$ defined by the expansions (10). We have $(-1, 1).(u, v, \tau) = (-u, -v, \tau)$ and $(1, -1).(u, v, \tau) = (u, v, -\tau)$, i.e., $G \times G$ acts separately on each coordinate. Thus, identifying $V_\varepsilon^j/G^2$ with the image of $\varphi \circ h_\varepsilon^j$ in $A$. Note that $B_\varepsilon^j = V_\varepsilon^j/G$ is smooth (except for a finite number of points) and the coherence of the $B_\varepsilon^j$ follows from the coherence of $V_\varepsilon^j$ and the action of $G$. Now by taking $B$ and by gluing on various varieties $B_\varepsilon^j\backslash\{\text{some points}\}$, we obtain a smooth complex manifold $\widehat{B}$ which is a double cover of the abelian variety $\widetilde{A}$ (constructed in proposition 2.3) ramified along $\mathcal{C}_1+\mathcal{C}_{-1}$, and therefore can be completed to an algebraic cyclic cover of $\widetilde{A}$. To see what happens to the missing points, we must investigate the image of $\Gamma\times\{0\}$ in $\cup B_\varepsilon^j$. The quotient $\Gamma\times\{0\}/G$ is birationally equivalent to the Riemann surface $\Upsilon$ of genus 7:

$$\begin{aligned}\Upsilon: \quad & \frac{65}{4}y^3+\frac{93}{64}x^3y^2+\frac{3}{8192}\left(-9829x^4+26\,112b_1\right)x^2y\\ &+x\left(-\frac{10\,299}{65\,536}x^8-\frac{123}{256}b_1x^4+b_2+\frac{1\,536\,298\,731}{52}\right)=0,\end{aligned}$$

where $y=uv, x=u^2$. The Riemann surface $\Upsilon$ is birationally equivalent to $\mathcal{C}$. The only points of $\Upsilon$ fixed under $(u,v)\mapsto(-u,-v)$ are the points at $\infty$, which correspond to the ramification points of the map $\Gamma\times\{0\}\overset{2-1}{\rightarrow}\Upsilon:(u,v)\mapsto(x,y)$ and coincides with the points at $\infty$ of the Riemann surface $\mathcal{C}$. Then the variety $\widehat{B}$ constructed above is birationally equivalent to the compactification $\overline{B}$ of the generic invariant surface $B$. So $\overline{B}$ is a cyclic double cover of the abelian surface $\widetilde{A}$ ramified along the divisor $\mathcal{C}_1+\mathcal{C}_{-1}$, where $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ have two points in commune at which they are tangent to each other. It follows that The system (8) is algebraic complete integrable in the generalized sense. Moreover, $\overline{B}$ is smooth except at the point lying over the singularity (of type $A_3$) of $\mathcal{C}_1+\mathcal{C}_{-1}$. In term of an appropriate local holomorphic coordinate system $(X,Y,Z)$, the local analytic equation about this singularity is $X^4+Y^2+Z^2=0$. Now, let $\widetilde{B}$ be the resolution of singularities of $\overline{B}$, $\mathcal{X}(\widetilde{B})$ be the Euler characteristic of $\widetilde{B}$ and $p_g(\widetilde{B})$ the geometric genus of $\widetilde{B}$. Then $\widetilde{B}$ is a surface of general type with invariants: $\mathcal{X}(\widetilde{B})=1$ and $p_g(\widetilde{B})=2$. This concludes the proof of the theorem.

**Remark 4.1.** *The asymptotic solution (52) can be read off from (31) and the change of variable:* $q_1=\sqrt{z_1}$, $q_2=z_2$, $p_1=z_4/q_1$, $p_2=z_3$. *The function* $z_1$ *has a simple pole along the divisor* $\mathcal{C}_1+\mathcal{C}_{-1}$ *and a double zero along a Riemann surface of genus 7 defining a double cover of* $\widetilde{A}$ *ramified along* $\mathcal{C}_1+\mathcal{C}_{-1}$.

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