Modern and classical aspects of integrable systems

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Preface

This book is devoted to topological and geometric aspects of the theory of classical and modern integrable systems. Integrable Hamiltonian systems are nonlinear ordinary differential equations described by a Hamiltonian function and possessing sufficiently many independent constants of motion in involution. The regular compact level manifolds defined by the intersection of the constants of motion are diffeomorphic to a real torus on which the motion is quasi-periodic. We study several approaches to integrable systems.

The problem of finding and integrating Hamiltonian systems, has attracted a considerable amount of attention in recent decades. Beside the fact that many integrable Hamiltonian systems have been on the subject of powerful and beautiful theories of mathematics, another motivation for its study is : the concepts of integrability which are applying to an increasing number of physical systems, biological phenomena, population dynamics, chemical rate equations, to mention only a few. However, it seems still hopeless to describe or even to recognize with any facility, those Hamiltonian systems which are integrable, though they are exceptional.

Last century, mechanics was dominated by the question whether a dynamical system can be solved by quadratures, i.e., by a finite number of algebraic operations including the inverting of functions. The classical approach to solving integrable systems was based on solving the Hamilton-Jacobi equation by separation of variables, after an appropriate change of coordinates; for every problem finding this transformation required a great deal of ingenuity. The solutions of these problems can be expressed in terms of theta functions related to Riemann surfaces. Some examples of this are : a) Jacobi's integration [104] of the geodesics on ellipsoid by using elliptic coordinates and various tricks. b) Neumann's study [189] of a mass point moving on the sphere under the influence of a linear force, using the spherical elliptic coordinates. *c*) The third case of integrability (beside the cases of Euler and Lagrange) of the motion of a rigid body about a fixed point in the presence of gravity discovered by Kowalewski [121] and which is solved in terms of hyperelliptic integrals. *d*) We mention also Kötter'solution [119, 120] by quadratures in terms of hyperelliptic integrals of the integrable Clebsch's [45] and Lyapunov-Steklov's [169, 218] cases of Kirchhoff's equations [109] describing the motion of a solid body in an ideal fluid, to mention only a few. These problems and many other examples are studied in detail in the various chapters of this book. It is well known that historically, the developments of mechanics and algebraic geometry (in particular the theory of Riemann surfaces) were closely intertwined. It must be emphasized that the classical approach to proving that a system is integrable by quadratures (in terms of hyperelliptic integrals) was something very unsystematic and required a great deal of luck and ingenuity. Jacobi himself was very much aware of this difficulty and in his famous [104] "*Vorlesungen über Dynamik*", in the context of geodesic flow on the ellipsoid (before introducing the elliptic coordinates), he wrote : "*Die Hauptschwierigkeit bei der Integration gegebener Differentialgleichungen scheint in der Einführung des richtigen Variablen zu bestehen, zu deren Auffindung es kein allgemeine Regel giebt. Man daher das ungekehrte Verfahren einschlagen und nach erlangter Kenntniss einer merkwürdigen Substitution die Probleme aufsuchen, bei welchen dieselbe mit Glück zu brauchen ist*" Finally, after Poincaré had recognized that most Hamiltonian systems are not completely integrable, the interest in this subject decreased for more than half a century.

The discovery by Gardner, Greene, Kruskal and Miura [67] that the Korteweg-de Vries (KdV) equation [116] :

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \qquad u(x,0) = u(x), x \in \mathbb{R}$$

could be integrated via inverse spectral methods has generated an enormous number of new ideas in the area of Hamiltonian completely integrable systems. Lax [127] showed that this equation is equivalent to the so-called Lax equation :

$$\frac{dA}{dt} = [A,B] \equiv AB - BA,$$

where A (Sturm-Liouville operator) and B are the differential operators in x :

$$A = -\frac{\partial^2}{\partial x^2} + u, \qquad B = 4\frac{\partial^3}{\partial x^3} - 3\left(u\frac{\partial}{\partial x} + \frac{\partial u}{\partial x}\right).$$

Lax equation means that, under the time evolution of the system, the linear operator $A(t)$ remains similar to $A(0)$. So the spectrum of A is conserved, i.e. it undergoes an isospectral deformation. The eigenvalues of A, viewed as functionals, represent the integrals (constants of the motion) of the KdV equation. Around 1974, Mc Kean-van Moerbeke [171], Dubrovin-Novikov [54] solved the periodic problem for the KdV equation (for $x \in S^1$) in terms of a linear motion on a real torus. This torus is the real part of the Jacobi variety of a hyperelliptic curve with branch points defined by the simple periodic and anti-periodic spectrum of A. Also the motion is a straight line in the variables of the well known Abel-Jacobi map. A parallel theory related to Jacobi matrices had its origin in the periodic Toda problem. The Toda lattice equations [223] (discretized version of the KdV equation) motion of n particles with exponential restoring forces are governed by the following Hamiltonian:

$$H = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{k=1}^{n} e^{q_k - q_{k+1}}.$$

Krichever [122] generalized these ideas to differential operators of any order, among which is the important Kadomtsev-Petviashvili (KP) equation [107] :

$$\frac{3}{4}\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - \frac{1}{4}\left(6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right)\right),$$

KdV equation and the equation of a non-linear string (Boussinesq equation [39]) :

$$3\frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\left(6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) = 0.$$

Also this theory was generalized to difference operators of any order by van Moerbeke and Mumford [234]. They worked out a systematic method which provides an algebraic map from the invariants manifolds defined by the intersection of the constants of the motion to the Jacobi variety of an algebraic curve associated to Lax equation.

As mentioned above, the resolution of the KdV equation has generated an enormous number of new ideas in the area of Hamiltonian completely integrable systems. It has led to unexpected connections between mechanics, spectral theory, Lie algebra theory, algebraic geometry and even differential geometry. All these connections have generated renewed interest in the questions around complete integrability of finite and infinite dimensional systems, ordinary and partial differential equations. However given a Hamiltonian system, it remains often hard to fit it into any of those general frameworks. But luckily, most of the problems possess the much richer structure of the so-called algebraic complete integrability (concept introduced et systematized by Adler and van Moerbeke). A dynamical system is algebraic completely integrable in the sense of Adler-van Moerbeke [12, 15, 17] if it can be linearized on a complex algebraic torus $\mathbf{C}^n/Lattice$ (=Abelian variety). The invariants (often called first integrals or constants) of the motion are polynomials and the phase space coordinates, or some algebraic functions of these, restricted to a complex invariant variety defined by putting these invariants equal to generic constants, are meromorphic functions on an Abelian variety. Moreover, in the coordinates of this Abelian variety, the flows (run with complex time) generated by the constants of the motion are straight lines. However, besides the fact that many Hamiltonian completely integrable systems posses this structure, another motivation for its study is due to the fact that algebraic completely integrable systems come up systematically whenever you study the isospectral deformation of some linear operator containing a rational indeterminate. Indeed a theorem of Adler-Kostant-Symes [6] applied to Kac-Moody algebras provides such systems which, by a theorem of van Moerbeke-Mumford [234], are algebraic completely integrable.

Strange as it may seem, the use of the Lax spectral curve technique may not give the tori correctly, but perhaps with period doubling, in contrast with the statement that the correct tori would be obtained by the Kowalewski-Painlevé analysis. This indicated a need for caution in interpretation of the result for tori calculated from the Lax spectral curve technique. A striking example of this phenomenon already appears in the Euler-equations associated to a class of geodesic flow on $SO(4)$ for a left-invariant diagonal metric. Haine [83] has given a proof using Kowalewski-Painlevé analysis that the four quadrics in this problem intersect in the affine part of an Abelian surface characterized as the Prym variety $Prym(C/C_0)$ where C is a genus 3 curve and is a double covering of an elliptic curve C_0. In [7, 8], Adler and van Moerbeke linearize the problem, using the method of isospectral deformations, on a 2-dimensional Prym variety $Prym(K/K_0)$ of another genus 3 curve over an elliptic curve K_0. However, the two Abelian surfaces $Prym(C/C_0)$ and $Prym(K/K_0)$ are not isomorphic but only isogenous, they intimately related in the precise sense that they are dual of each other. It turns out that these two Abelian surfaces are isomorphic only up to multiplication of some (not all) periods by 2. The natural coordinates are not meromorphic on the tori but only their squares, if one linearizes the problem via Lax spectral curve technique. In contrast these natural coordinates are meromorphic on the invariant surfaces and on their natural completion into tori using the Kowalewski-Painlevé analysis. The relation between the two sets of tori is as follows: one set can be obtained from the other by doubling some but not all periods. The relationship between the curves C and K is complicated : $Prym(K/K_0)\backslash\Pi = \Theta \cap Prym(K/K_0) = C$, where Θ is a translate of the Riemann's theta divisor on $Jac(K)$ et $\Pi \subset Prym(K/K_0)$ is a Zariski-open set and $\Theta \cap Prym^*(K/K_0) = \Theta \cap Prym(C/C_0) = K$, where Θ is a translate of the Riemann's theta divisor on $Jac(C)$. Moser [180] was aware of a similar situation in the context of the Jacobi's geodesic flow problem on ellipsoids. Also some interesting integrable systems appear as coverings of algebraic completely integrable systems. The invariant varieties are coverings of Abelian varieties and these systems are called algebraic completely integrable in the generalized sense.

The concept of algebraic complete integrability is quite effective in small dimensions and has the advantage to lead to global results, unlike the existing criteria for real analytic integrability, which, at this stage are perturbation results. In fact, the overwhelming majority of dynamical systems, Hamiltonian or not, are non-integrable and possess regimes of chaotic behavior in phase space. The methods used are primarily analytical but heavily inspired by algebraic geometrical methods. Abelian varieties and cyclic coverings of Abelian varieties, very heavily studied by algebraic geometers, enjoy certain algebraic properties which can be translated into differential equations and their Laurent solutions.

In recent years, other important results have been obtained following studies on the KP and KdV hierarchies. The use of tau functions related to infinite dimensional Grassmannians, Fay identities, vertex operators and the Hirota's bilinear formalism led to obtaining remarkable properties concerning these algebras of infinite order differential operators as for example the existence of an infinite family of first integrals functionally independent and in involution. The elaboration of powerful methods and the discovery of their common algebraic structures led to important developments concerning the study of nonlinear problems (see for example [233] for an overview). The functions $\tau(t)$ are specific functions of time, constructed from sections of a determinant bundle on an infinite-dimensional Grassmannian manifold. These functions generalize the Riemann theta functions and they are solutions of the KP hierarchy, i.e, solutions of an infinite series of nonlinear partial differential equations [42] connecting infinity of functions of infinity variables. The functions $\tau(t)$ can be Schur polynomials, falling within Fredholm's group representation theory or determinants. Recently, a new type of tau function has appeared, within the framework of quantum gauge theory with gauge group $SU(N)$ when N is large. This led to the so-called matrix models (quantum gravity) for counting triangulations on certain surfaces (topology). The underlying models have remained relatively intractable except in two space-time dimensions; although being physically toy models, their structure is still very rich. The first tau function was introduced by Sato, Miwa and Jimbo in relation to the theory of isomonodromic deformations. It has been defined as a correlation function of certain quantum fields associated with the poles of a Fuchsian system on the Riemann sphere. These functions give information on the topology of moduli spaces of Riemann surfaces and are closely related to the theory of representations of Virasoro algebras and W-algebras. The $\tau(t)$ functions play an important role in a large number of branches of mathematics and theoretical physics, such as integrable systems, string theories, quantum-gauge theories, isomonodromic deformations, matrix models (quantum gravity), the associated matrix integrals which have power series expansions (perturbative series) and whose terms count the triangulations on surfaces (Feynman graphs), the module problems and in many other domains.

In fact in recent years, many problems related to algebraic geometry, combinatorics, probabilities and quantum gauge theory,..., have been solved explicitly by methods inspired by techniques from the study of integrable systems. In particular, the study of random matrices, a domain that establishes links with several problems, for example with combinatorics, probabilities, number theory, models of growth and random tailings and questions of communication technology. The functions $\tau(t)$

are the source of inspiration for many mathematicians and physicists in search of new algebraic structures appearing in mathematics and physics. The vertex operators give a good device to the investigation of the matrix models and the spectrum of the stochastic matrices.

Having said all that, it is perhaps time to pass to the contents of this book. It contains five large chapters each consisting of several sections and some appendices. We shall take a slightly more detailed look at the material included, knowing that the technical detail will be given at the level of each chapter. Also note that I shall not discuss in this book the weaker notion of analytic integrability; the perturbation techniques developed in that context are of a totally different nature. I have omitted too the topological and physical aspects of matrix models. In such a vast subject, one had to make a choice: the one presented here is interesting from several points of view. In short, in any case, many interesting things remain to discuss and especially to discover.

In chapter 1 we study the symplectic manifolds, which allow us, among other things, to introduce the Hamiltonian vector fields. It's firstly devoted to the study of combined adjoint and coadjoint orbits of a Lie group with an application in the case of the special orthogonal group $SO(n)$. Some properties on the Lie derivative and the inner product are also studied. Before proceeding to the actual study of symplectic manifolds, we begin by briefly recalling some notions about symplectic vector spaces and we develop the explicit calculation of symplectic structures on a differentiable manifold. A part is dedicated to the study of a central theorem of symplectic geometry namely Darboux's theorem: the symplectic manifolds (M, ω) of dimension $2m$ are locally isomorphic to $(\mathbb{R}^{2m}, \omega)$. The classic proof given by Darboux of his theorem is by recurrence on the dimension of the variety. We give a preview and see another demonstration due to Weinstein based on a result of Moser. An important section is assigned to the Hamiltonian vector fields. The latter form a Lie subalgebra of the space vector field and we show that the matrix associated with a Hamiltonian system forms a symplectic structure. Several properties concerning Hamiltonian vector fields, their connection with symplectic manifolds, Poisson manifolds or Hamiltonian manifolds as well as interesting examples are studied. We will see how to define a symplectic structure on the orbit of the coadjoint representation of a Lie group. The remainder is dedicated to the explicit computation of symplectic structures on adjoint and coadjoint orbits of a Lie group with particular attention given to the groups $SO(3)$ and $SO(4)$. As mentioned above, integrable Hamiltonian systems are nonlinear ordinary differential equations described by a Hamiltonian function and possessing sufficiently many independent constants of motion in involution. By the Arnold-Liouville theorem, the regular compact level manifolds defined by the intersection of the constants of motion are diffeomorphic to a real torus on which the motion is quasi-periodic as a consequence of the following differential geometric fact; a compact and connected n-dimensional manifold on which there exist n vector fields which commute and are independent at every point is diffeomorphic to an n-dimensional real torus and there is a transformation to so-called action-angle variables, mapping the flow into a straight line motion on that torus. We give a proof as direct as possible of the Arnold-Liouville theorem and we make a careful study of its connection with the concept of completely integrable systems and finally apply it to concrete situations. Many problems are studied in detail : the problem of the rotation of a rigid body about a fixed point, the problem of motion of a solid in an ideal fluid and Yang-Mills field with gauge group $SU(2)$.

In chapter 2, we give a detailed study of the integrable systems which can be written as Lax equations with a spectral parameter. Such equations have no a priori Hamiltonian content. However, through the Adler-Kostant-Symes (AKS) construction, we can produce Hamiltonian systems on coadjoint orbits in the dual space to a Lie algebra whose equations of motion take the Lax form. We outline an algebraic-geometric interpretation of the flows of these systems, which are shown to describe linear motion on a complex torus. The relationship between spectral theory and completely integrable systems is a fundamental aspect of the modern theory of integrable systems. This chapter surveys a number of classical and recent results and our purpose here is to give a motivated and a sketchy overview of this interesting subject. We also present a Lie algebra theoretical schema leading to integrable systems, based on the Kostant-Kirillov coadjoint action. Many problems on Kostant-Kirillov coadjoint orbits in subalgebras of infinite dimensional Lie algebras (Kac-Moody Lie algebras) yield large classes of extended Lax pairs. A general statement leading to such situations is given by the Adler-Kostant-Symes theorem and the van Moerbeke-Mumford linearization method provides an algebraic map from the complex invariant manifolds of these systems to the Jacobi variety (or some subabelian variety of it) of the spectral curve. The complex flows generated by the constants of the motion are straight line motions on these varieties. We study the isospectral deformation of periodic Jacobi matrices and general difference operators from an algebraic geometrical point of view and their relation with the Kac-Moody extension of some algebras. We present in detail the Griffith's approach and his cohomological interpretation of linearization test for solving integrable systems without reference to Kac-Moody algebras. His method is based on the observation that the tangent space to any deformation lies in a suitable cohomology group and on algebraic curves, higher cohomology can always be eliminated using duality theory. We explain how results from deformation theory and algebraic geometry can be used to obtain insight into the dynamics of integrable systems. These conditions are cohomological and the Lax equations turn out to have a natural cohomological interpretation. We discuss several examples of integrable systems of relevance in mathematical physics : geodesic flow on $SO(n)$, the Euler rigid body motion, the Manakov geodesic flow on the group $SO(4)$, Jacobi geodesic flow on an ellipsoid and Neumann problem, the Lagrange top, a family of integrable systems, the coupled nonlinear Schrödinger equations, the Yang-Mills equations, the Kowalewski spinning top, the Goryachev-Chaplygin top, the Toda lattice, Nahm's equations and the n-dimensional rigid body .

The aim of chapter 3 is to present an overview of the active area of algebraic completely integrable systems in the sense of Adler and van Moerbeke. These are integrable systems whose trajectories are straight line motions on Abelian varieties (complex algebraic tori). We make, via the Kowalewski-Painlevé analysis, a study of the level manifolds of the systems, which are described explicitly as being affine part of Abelian varieties and the flow can be solved by quadrature, that is to say their solutions can be expressed in terms of Abelian integrals. We also describe an explicit embedding of these Abelian varieties which complete the generic invariant surfaces, into projective spaces. The Adler-van Moerbeke method's which will be used is devoted to illustrate how to decide about the algebraic completely integrable Hamiltonian systems and it is primarily analytical but heavily inspired by algebraic geometrical methods. Many problems are studied in detail : Euler top, a five-dimensional system, the Hénon-Heiles system, the Kowalewski top, the Manakov geodesic flow on the group $SO(4)$, the Adler-van Moerbeke geodesic flow on $SO(4)$ with a quartic invariant, the geodesic flow on

$SO(n)$ for a left invariant metric, a family of integrable systems, the periodic 5-particle Kac-van Moerbeke lattice, generalized periodic Toda systems, the Gross-Neveu system and the Kolossof potential .

In chapter 4, we tackle the study of generalized algebraic completely integrable systems. There are many examples of differential equations which have the weak Painlevé property that all movable singularities of the general solution have only a finite number of branches and some interesting integrable systems appear as coverings of algebraic completely integrable systems. The invariant varieties are coverings of Abelian varieties and these systems are called algebraic completely integrable in the generalized sense. These systems are Liouville integrable and by the Arnold-Liouville theorem, the compact connected manifolds invariant by the real flows are tori; the real parts of complex affine coverings of Abelian varieties. Most of these systems of differential equations possess solutions which are Laurent series of $t^{1/n}$ (t being complex time) and whose coefficients depend rationally on certain algebraic parameters. We discuss some interesting and well known examples of algebraic completely integrable systems : a 4-dimensional algebraically integrable system in the generalized sense as part of a 5-dimensional algebraically integrable system, the Hénon-Heiles and a 5-dimensional system, the RDG potential and a 5-dimensional system, the Goryachev-Chaplygin top and a 7-dimensional system, the Lagrange top, the (generalized) Yang-Mills system and cyclic covering of Abelian varieties.

Chapter 5 covers in detail : the KdV equation and the inverse scattering method (based on Schrödinger and Gelfand-Levitan equations) used to solve it exactly. We study some generalities on the algebra of infinite order differential operators. The algebras of Virasoro, Heisenberg, nonlinear evolution equations such as the KdV, Boussinesq and KP equations play a crucial role in this study. We make a careful study of some connection between pseudo-differential operators, symplectic structures, KP hierarchy and tau functions based on the Sato-Date-Jimbo-Miwa-Kashiwara theory. A few other connections and ideas concerning the KdV and Boussinesq equations, the Gelfand-Dickey flows, the Heisenberg and Virasoro algebras are given. The study of the KP and KdV hierarchies, the use of tau functions related to infinite dimensional Grassmannians, Fay identities, vertex operators and the Hirota's bilinear formalism led to obtaining remarkable properties concerning these algebras as for example the existence of an infinite family of first integrals functionally independent and in involution.

The book is supported by some appendices which contain some basis concepts, both for their own interests and because they leads to the clarification of a number of results obtained in this book. It is preferable to pass them on first reading and to confine themselves to consulting them at the moment of use.

Appendix A deals with the study of some notions concerning the variational principle. We will establish the Euler-Lagrange differential equation, the Hamilton's canonical equations, the Hamilton-Jacobi partial differential equation and explain how it is widely used in practice to solve some problems. As an application, we study the geodesics, the harmonic oscillator as well as the Kepler problem.

Appendix B presents the basic ideas and properties of elliptic functions and elliptic integrals as an expository essay. It explores some of their numerous consequences and includes an application to the study of the simple pendulum.

In appendix C, we collect some notions on compact Riemann surfaces because of their importance for integrable systems.

In appendix D, we give some results about Abelian surfaces which are used, as well as the basic techniques to study two-dimensional algebraic completely integrable systems.

Appendix E is devoted to the construction of holomorphic differential forms on compact Riemann surfaces. When the Riemann surface is concretely described, we show that one can usually present the basis of holomorphic differentials explicitly.

Appendix F provides a short and quick exposition of some important aspects of meromorphic theta functions for compact Riemann surfaces. The study of theta functions is done via an analytical approach using meromorphic functions in the framework of Mumford.

At the end of this book, it is worth to mention some similar problems. Abelian varieties, very heavily studied by algebraic geometers, enjoy certain algebraic properties which can then be translated into differential equations and their Laurent solutions. Among the results presented in this book, there is an explicit calculation of invariants for Hamiltonian systems which cut out an open set in an Abelian variety and various algebraic curves related to these systems are given explicitly. The integrable systems presented here are interesting problems, particular to experts of Abelian varieties who may want to see explicit examples of a correspondence for varieties defined by different algebraic curves. The methods used are primarily analytical but heavily inspired by algebraic geometrical methods.

The book is intended for a wide readership of mathematicians and physicists, students pursuing graduate, masters and higher degrees in mathematics and mathematical physics. It is devoted to topological and geometric aspects of the theory of classical and modern integrable systems (subject of great interest). It also contains contemporary integrability results discovered in the last few decades and are used in different domains of physics and mathematics. Many examples and exercises with solutions are provided throughout the text. At the end we include a bibliography with some basic books and papers easily accessible and a detailed index allowing the reader to quickly locate an element treated in the book.

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