

On Poincaré lemma or Volterra theorem about differential forms and cohomology groups

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Abstract. The Poincaré lemma (or Volterra theorem) is of utmost importance both in theory and in practice. It tells us every differential form which is closed, is locally exact. In other words, on a contractible manifold all closed forms are exact. The aim of this paper is to present some direct proofs of this lemma and explore some of its numerous consequences. Some connections with Čech-De Rham-Dolbeault cohomologies, $\bar{\partial}$ -Poincaré lemma or Dolbeault-Grothendieck lemma are given.

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1 Introduction

Let ω be a k -differential form ($k > 0$) on an open subset $U \subset \mathbb{R}^n$. We assume that ω is \mathcal{C}^1 . Recall that ω is closed (or is a cocycle) if $d\omega = 0$. Similarly, ω is exact (or cohomological at 0) if there is a \mathcal{C}^1 , $(k-1)$ -differential form λ on U such that $\omega = d\lambda$. Notice that there are no exact 0-forms because there are no -1 -forms. It is well known that every exact differential form is closed and that the reciprocal is false in general. Poincaré lemma ensures that it is true on a star-shaped open subset of \mathbb{R}^n whose definition is as follows : let $[a, b] = \{\lambda a + (1 - \lambda)b, \lambda \in [0, 1]\}$, $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$. We say that an open subset $U \subset \mathbb{R}^n$ is star-shaped if there exists an a in U , such that for all b in U , $[a, b] \subset U$. In other words, if the line segment from b to a is in U .

We will study Poincaré lemma which is considered fundamental both in theory and in practice. Already at the university level and as evidenced by several scientific books, this is a key result for the study of many problems in mathematics, physics and the theorem itself has applications in areas ranging

from electrodynamics to differential and integral calculus on varieties. At a higher level it intervenes for example when studying the cohomology of De Rham varieties [7] to name only this striking example. Let's note for information that the so currently called Poincaré lemma is due to Volterra. Indeed, the Poincaré lemma is really Volterra theorem ; the work of Volterra is contained in several notes published in the Rendiconti of the Accademia dei Lincei [10] (see also [6]). In this work we use as everyone the name Poincaré lemma instead of Volterra theorem and we leave the question to be clarified by historians.

The lemma to be demonstrated is a typical example of a local result, so it suffices to prove it in local coordinates, for example in an arbitrarily small open of \mathbb{R}^n , but it must be admitted that the technical details are much more complicated to demonstrate when we move from the case of 1-forms to k -differential forms. All known proofs of this lemma, using these coordinates are often too computerized and rarely illuminating.

We offer some proofs of the Poincaré lemma in this paper because the proofs represent widely different views of the subject. The first proof is given in a very classical setting and represents the classical point of view ; it is both elementary and constructive but the problem is that it is a bit long and technical. Then, our problem is to give a quick proof of this lemma although requiring certain knowledges pushed in differential geometry ; the proofs uses very modern machinery and represent a more modern point of view. Both of these points of view have merit and so we demonstrate them both. The paper is divided in some sections and subsections, each of them devoted to various and complementary aspects of the problem concerning, in particular, connections with Cech-De Rham-Dolbeault cohomologies, $\bar{\partial}$ -Poincaré lemma or Dolbeault-Grothendieck lemma.

2 The Poincaré lemma

We give a little information (which will be needed in the second proof) about one-parameter group of diffeomorphisms, differential operators, Lie derivative, inner product and Cartan's formula. This discussion is brief, but should be enough to define notation. Let M be a differentiable manifold of dimension m . Let TM be the tangent bundle to M , i.e., the union of spaces tangent to M at all points x , $TM = \bigcup_{x \in M} T_x M$. This bundle has a natural structure of differentiable variety of dimension $2m$ and it allows us to convey immutably to the manifolds the whole theory of ordinary differential equations. A vector field (we also say section of the tangent bundle) on M is an application, denoted X , which at every point $x \in M$ associates a tangent vector $X_x \in T_x M$. In other words, $X : M \rightarrow TM$, is an application such that if $\pi : TM \rightarrow M$, is the natural projection, we have $\pi \circ X = id_M$. Let (x_1, \dots, x_m) be a local coordinate system in a neighborhood $U \subset M$. In this

system the vector field X is written in the form

$$X = \sum_{k=1}^m f_k(x) \frac{\partial}{\partial x_k}, \quad x \in U,$$

where the functions $f_1, \dots, f_m : U \rightarrow \mathbb{R}$, are the components of X with respect to (x_1, \dots, x_m) . A vector field X is differentiable if its components $f_k(x)$ are differentiable functions. Given a point $x \in M$, we write $g_t^X(x)$ (or simply $g_t(x)$) the position of x after a displacement of a duration $t \in \mathbb{R}$. There is thus an application $g_t^X : M \rightarrow M$, $t \in \mathbb{R}$, which is a diffeomorphism (a one-to-one differentiable mapping with a differentiable inverse), by virtue of the theory of differential equations. The vector field X generates a one-parameter group of diffeomorphisms g_t^X on M , i.e., a differentiable application (\mathcal{C}^∞) : $M \times \mathbb{R} \rightarrow M$, satisfying a group law :

i) $\forall t \in \mathbb{R}$, $g_t^X : M \rightarrow M$ is a diffeomorphism.

ii) $\forall t, s \in \mathbb{R}$, $g_{t+s}^X = g_t^X \circ g_s^X$.

The condition *ii)* means that the mapping $t \mapsto g_t^X$, is a homomorphism of the additive group \mathbb{R} into the group of diffeomorphisms of M . It implies that

$$g_{-t}^X = (g_t^X)^{-1},$$

because $g_0^X = id_M$ is the identical transformation that leaves every point invariant. The one-parameter group of diffeomorphisms g_t^X on M , which we have just described is called a flow and it admits the vector field X for velocity fields

$$\frac{d}{dt} g_t^X(x) = X(g_t^X(x)),$$

with the initial condition : $g_0^X(x) = x$. Obviously

$$\left. \frac{d}{dt} g_t^X(x) \right|_{t=0} = X(x).$$

Hence by these formulas $g_t^X(x)$ is the curve on the manifold that passes through x and such that the tangent at each point is the vector $X(g_t^X(x))$. The vector field X generates a unique group of diffeomorphisms of M . With every vector field X we associate the first-order differential operator L_X . This is the differentiation of functions in the direction of the vector field X . We have

$$L_X : \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M), \quad F \mapsto L_X F,$$

where

$$L_X F(x) = \left. \frac{d}{dt} F(g_t^X(x)) \right|_{t=0}, \quad x \in M.$$

$\mathcal{C}^\infty(M)$ being the set of functions $F : M \rightarrow \mathbb{R}$, of class \mathcal{C}^∞ . The operator L_X is linear : $L_X(\alpha_1 F_1 + \alpha_2 F_2) = \alpha_1 L_X F_1 + \alpha_2 L_X F_2$, where $\alpha_1, \alpha_2 \in \mathbb{R}$, and satisfies the Leibniz formula :

$$L_X(F_1 F_2) = F_1 L_X F_2 + F_2 L_X F_1.$$

Since $L_X F(x)$ only depends on the values of F in the neighborhood of x , we can apply the operator L_X to the functions defined only in the neighborhood of a point, without the need to extend them to the full variety M . Let (x_1, \dots, x_m) be a local coordinate system on M . In this system the vector field X has components f_1, \dots, f_m and the flow g_t^X is defined by a system of differential equations. Therefore, the derivative of the function $F = F(x_1, \dots, x_m)$ in the direction of X is written

$$L_X F = f_1 \frac{\partial F}{\partial x_1} + \dots + f_m \frac{\partial F}{\partial x_m}.$$

In other words, in the coordinates (x_1, \dots, x_m) the operator L_X is written $L_X = f_1 \frac{\partial}{\partial x_1} + \dots + f_m \frac{\partial}{\partial x_m}$, this is the general form of the first order linear differential operator.

Let M and N be two differentiable manifolds of dimension m and n respectively, $U \subset M$, $V \subset N$ two open subsets. For any differentiable application $g : U \rightarrow V$, and any k -differential form in V ,

$$\omega = \sum_{1 \leq i_1, \dots, i_k \leq n} f_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

we define a k -differential form in U (called the pull-back by g or inverse image or the transpose of ω by g) by setting

$$g^* \omega = \sum_{1 \leq i_1, \dots, i_k \leq n} (f_{i_1, \dots, i_k} \circ g) dg_{i_1} \wedge \dots \wedge dg_{i_k},$$

where

$$dg_{i_l} = \sum_{j=1}^m \frac{\partial g_{i_l}}{\partial y_j} dy_j,$$

are 1-forms in U . Note that g^* is a linear operator from the space of k -forms on N to the space of k -forms on U (the asterisk indicates that g^* operates in the opposite direction of g). Let X be a vector field on a differentiable manifold M . We recalled above that the vector field X generates a unique group of diffeomorphisms g_t^X (that we also note g_t) on M , solution of the differential equation

$$\frac{d}{dt} g_t^X(p) = X(g_t^X(p)), \quad p \in M,$$

with the initial condition $g_0^X(p) = p$. Let ω be a k -differential form. The Lie derivative of ω with respect to X is the k -form differential defined by

$$L_X \omega = \left. \frac{d}{dt} g_t^* \omega \right|_{t=0} = \lim_{t \rightarrow 0} \frac{g_t^*(\omega(g_t(p))) - \omega(p)}{t}.$$

In general, for $t \neq 0$, we have

$$\frac{d}{dt}g_t^*\omega = \frac{d}{ds}g_{t+s}^*\omega \Big|_{s=0} = g_t^* \frac{d}{ds}g_s^*\omega \Big|_{s=0} = g_t^*(L_X\omega). \quad (1)$$

It is easily verified that for the k -differential form $\omega(g_t(p))$ at the point $g_t(p)$, the expression $g_t^*\omega(g_t(p))$ is indeed a k -differential form in p . For all $t \in \mathbb{R}$, the application $g_t : \mathbb{R} \rightarrow \mathbb{R}$ being a diffeomorphism then dg_t and dg_{-t} are applications,

$$\begin{aligned} dg_t & : T_pM \longrightarrow T_{g_t(p)}M, \\ dg_{-t} & : T_{g_t(p)}M \longrightarrow T_pM. \end{aligned}$$

The Lie derivative of a vector field Y in the direction X is defined by

$$L_X Y = \frac{d}{dt}g_{-t}Y \Big|_{t=0} = \lim_{t \rightarrow 0} \frac{g_{-t}(Y(g_t(p))) - Y(p)}{t}.$$

In general, for $t \neq 0$, we have

$$\frac{d}{dt}g_{-t}Y = \frac{d}{ds}g_{-t-s}Y \Big|_{s=0} = g_{-t} \frac{d}{ds}g_{-s}Y \Big|_{s=0} = g_{-t}(L_Y).$$

An interesting operation on differential forms is the inner product that is defined as follows : the inner product of a k -differential form ω by a vector field X on a differentiable manifold M is a $(k-1)$ -differential form, denoted $i_X\omega$, defined by

$$(i_X\omega)(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}),$$

where X_1, \dots, X_{k-1} are vector fields. If

$$X = \sum_{j=1}^m X_j(x) \frac{\partial}{\partial x_j},$$

is the local expression of the vector field on the variety M of dimension m and

$$\omega = \sum_{i_1 < i_2 < \dots < i_k} f_{i_1 \dots i_k}(x) dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

then, the expression of the inner product in local coordinates is given by

$$i_{\frac{\partial}{\partial x_j}} \omega = \frac{\partial}{\partial(dx_j)} \omega,$$

where we put dx_j in first position in ω . Moreover, the well-known Cartan homotopy formula is fundamental to the Lie derivative and can be used as a definition :

$$L_X\omega = d(i_X\omega) + i_X(d\omega). \quad (2)$$

For a differential form ω , we have

$$i_X L_X\omega = L_X i_X\omega.$$

Theorem 1 *On a star-shaped open subset U of \mathbb{R}^n , any closed differential form is exact.*

Proof 1 : a) (For a 1-differential form). Let $U \subset \mathbb{R}^n$, be a star-shaped open subset relative to one of its points that we note a . Let

$$\omega = \sum_{i=1}^n f_i dx_i$$

, be a closed 1-differential form in U , and we will show that it is exact. In other words, that there is a 0-differential form in U , i.e., a \mathcal{C}^1 application $h : U \rightarrow \mathbb{R}$ such that $\omega = dh$ or what amounts to the same, such that :

$$f_i = \frac{\partial h}{\partial x_i}.$$

(Indeed, if ω is exact, then by definition

$$\omega = \sum_{i=1}^n f_i dx_i = dh = \sum_{i=1}^n \frac{\partial h}{\partial x_i} dx_i,$$

hence the result). We will see that for the regularity of this 1-form, h is \mathcal{C}^2 . By hypothesis, U is a star-shaped open subset and according to the fundamental theorem of differential and integral calculus as well as that of derivation of composite functions, we have

$$\begin{aligned} h(x) - h(a) &= \int_0^1 \frac{d}{dt} h(a + t(x - a)) dt, \\ &= \int_0^1 \sum_{j=1}^n \frac{\partial h}{\partial x_j}(a + t(x - a)) \cdot (x_j - a_j) dt. \end{aligned}$$

A function h satisfying the relation $f_i = \frac{\partial h}{\partial x_i}$, is only defined to an additive constant. We can therefore put

$$h(x) = \int_0^1 \sum_{j=1}^n \frac{\partial h}{\partial x_j}(a + t(x - a)) \cdot (x_j - a_j) dt.$$

The application h is well defined on U because for all $t \in [0, 1]$, $a + t(x - a) \in U$ (U is a star-shaped open subset). Notice that $\frac{\partial h}{\partial x_i}$ exist; it is a function defined by an integral. We have

$$\begin{aligned} \frac{\partial h}{\partial x_i}(x) &= \int_0^1 \frac{\partial}{\partial x_i} \left(\sum_{j=1}^n f_j(a + t(x - a)) \cdot (x_j - a_j) \right) dt, \\ &= \int_0^1 \sum_{j=1}^n \left(\frac{\partial f_j}{\partial x_i}(a + t(x - a)) \cdot t \cdot (x_j - a_j) + f_j(a + t(x - a)) \cdot \delta_{ij} \right) dt, \end{aligned}$$

where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$. Since w is closed, then

$$d\omega = \sum_{i=1}^3 df_i \wedge dx_i = \sum_{\substack{1 \leq i, j \leq 3 \\ i < j}} \left(\frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} \right) dx_i \wedge dx_j.$$

As a result, ω is closed (i.e., $d\omega = 0$) if and only if

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i},$$

because $dx_i \wedge dx_j \neq 0$, $i \neq j$. If $i = j$, the relations in question are trivial and for $i > j$, it is obviously enough to swap the indices i and j . Then,

$$\begin{aligned} \frac{\partial h}{\partial x_i}(x) &= \int_0^1 \left(t \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}(a + t(x - a))(x_j - a_j) + f_i(a + t(x - a)) \right) dt, \\ &= \int_0^1 \left(t \frac{d}{dt} f_i(a + t(x - a)) + f_i(a + t(x - a)) \right) dt, \\ &= \int_0^1 \frac{d}{dt} (t f_i(a + t(x - a))) dt, \\ &= f_i(x), \end{aligned}$$

where $1 \leq i \leq n$ and $x \in U$. The 0-differential forms in U , i.e., continuous applications $U \rightarrow \mathbb{R}$ and since $\omega = \sum_{i=1}^n f_i dx_i$, the applications f_i are \mathcal{C}^1 and

we deduce from the relations above that h is \mathcal{C}^2 on U .

b) (For a k -differential form, $k \geq 2$). Without restricting the generality, we suppose that U is a star-shaped open subset with respect to 0. Let ψ be the application from the set of l -differential forms on U to the set of $(l - 1)$ -differential forms on U defined by

$$\psi(\lambda) = \sum_{1 \leq i_1, \dots, i_k \leq n} \sum_{j=1}^l (-1)^{j-1} \left(\int_0^1 t^{l-1} g_{i_1, \dots, i_l}(t) dt \right) \pi_{i_j} dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_l},$$

where

$$\begin{aligned} \lambda &= \sum_{1 \leq i_1, \dots, i_l \leq n} g_{i_1, \dots, i_l} dx_{i_1} \wedge \dots \wedge dx_{i_l}, \\ g_{i_1, \dots, i_l}(t) &: U \rightarrow \mathbb{R}, \quad x \mapsto g_{i_1, \dots, i_l}(tx), \quad t \in [0, 1], \end{aligned}$$

the application π_{i_j} is the projection on the i_j^{th} coordinate and $\widehat{dx_{i_j}}$ denotes the term omitted. Let

$$\omega = \sum_{1 \leq i_1, \dots, i_k \leq n} f_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

be a \mathcal{C}^1 , k -differential form in U . The idea of the proof is to show that

$$\omega = \psi(d\omega) + d(\psi(\omega)),$$

because in this case since by hypothesis ω is closed, then $\omega = d(\psi(\omega))$ (by construction, $\psi(0) = 0$) and therefore, the form ω is exact (note that $\psi(\omega)$ is \mathcal{C}^1 if ω is). We have

$$d\omega = \sum_{1 \leq i_1, \dots, i_k \leq n} \frac{\partial f_{i_1, \dots, i_k}}{\partial x_i} \wedge dx_i \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k},$$

and

$$\begin{aligned} & \psi(d\omega) \\ &= \sum_{i=1}^n \sum_{1 \leq i_1, \dots, i_k \leq n} \left[\left(\int_0^1 t^k \frac{\partial f_{i_1, \dots, i_k}}{\partial x_i}(t) dt \right) p_i dx_{i_1} \wedge \dots \wedge dx_{i_k} \right. \\ & \quad \left. + \sum_{j=1}^k (-1)^j \left(\int_0^1 t^k \frac{\partial f_{i_1, \dots, i_k}}{\partial x_i}(t) dt \right) p_{i_j} dx_i \wedge dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_k} \right], \end{aligned}$$

or

$$\begin{aligned} & \psi(d\omega) \\ &= \sum_{1 \leq i_1, \dots, i_k \leq n} \left[\sum_{i=1}^n \left(\int_0^1 t^k \frac{\partial f_{i_1, \dots, i_k}}{\partial x_i}(t) dt \right) p_i dx_{i_1} \wedge \dots \wedge dx_{i_k} \right] \\ & \quad - \sum_{1 \leq i_1, \dots, i_k \leq n} \left[\sum_{i=1}^n \sum_{j=1}^k (-1)^{j-1} \left(\int_0^1 t^k \frac{\partial f_{i_1, \dots, i_k}}{\partial x_i}(t) dt \right) \right. \\ & \quad \left. p_{i_j} dx_i \wedge dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_k} \right]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & d(\psi(\omega)) \\ &= \sum_{1 \leq i_1, \dots, i_k \leq n} \sum_{j=1}^k (-1)^{j-1} d \left[\left(\int_0^1 t^{k-1} f_{i_1, \dots, i_k}(t) dt \right) p_{i_j} \right] \\ & \quad \wedge dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_k}, \end{aligned}$$

or

$$\begin{aligned} & d(\psi(\omega)) \\ &= \sum_{1 \leq i_1, \dots, i_k \leq n} \sum_{j=1}^k (-1)^{j-1} \sum_{i=1}^n \left[\left(\int_0^1 t^{k-1} \frac{\partial f_{i_1, \dots, i_k}}{\partial x_i}(t) dt \right) p_{i_j} \right. \\ & \quad \left. + \left(\int_0^1 t^{k-1} f_{i_1, \dots, i_k}(t) dt \right) \delta_{i_j, i} \right] dx_i \wedge dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_k}, \end{aligned}$$

or

$$\begin{aligned}
& d(\psi(\omega)) \\
&= \sum_{1 \leq i_1, \dots, i_k \leq n} \sum_{i=1}^n \sum_{j=1}^k (-1)^{j-1} \left(\int_0^1 t^k \frac{\partial f_{i_1, \dots, i_k}(t)}{\partial x_i} dt \right) \\
&\quad p_{i_j} dx_i \wedge dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_k} \\
&+ \sum_{1 \leq i_1, \dots, i_k \leq n} \sum_{j=1}^k (-1)^{j-1} \left(\int_0^1 t^{k-1} f_{i_1, \dots, i_k}(t) dt \right) \\
&\quad dx_{i_j} \wedge dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_k},
\end{aligned}$$

and finally,

$$\begin{aligned}
& d(\psi(\omega)) \\
&= \sum_{1 \leq i_1, \dots, i_k \leq n} \sum_{i=1}^n \sum_{j=1}^k (-1)^{j-1} \left(\int_0^1 t^k \frac{\partial f_{i_1, \dots, i_k}(t)}{\partial x_i} dt \right) \\
&\quad p_{i_j} dx_i \wedge dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_j}} \wedge \dots \wedge dx_{i_k} \\
&+ k \sum_{1 \leq i_1, \dots, i_k \leq n} \left(\int_0^1 t^{k-1} f_{i_1, \dots, i_k}(t) dt \right) dx_{i_1} \wedge \dots \wedge dx_{i_k}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \psi(d\omega) + d(\psi(\omega)) \\
&= \sum_{1 \leq i_1, \dots, i_k \leq n} \sum_{i=1}^n \left(\int_0^1 t^k \frac{\partial f_{i_1, \dots, i_k}(t)}{\partial x_i} dt \right) p_i dx_{i_1} \wedge \dots \wedge dx_{i_k} \\
&\quad + \sum_{1 \leq i_1, \dots, i_k \leq n} \left(\int_0^1 k t^{k-1} f_{i_1, \dots, i_k}(t) dt \right) dx_{i_1} \wedge \dots \wedge dx_{i_k}, \\
&= \sum_{1 \leq i_1, \dots, i_k \leq n} \left(\int_0^1 \frac{d}{dt} \left(t^k f_{i_1, \dots, i_k}(t) \right) dt \right) dx_{i_1} \wedge \dots \wedge dx_{i_k}, \\
&= \sum_{1 \leq i_1, \dots, i_k \leq n} f_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}, \\
&= \omega,
\end{aligned}$$

and proof 1 ends.

Proof 2 : Using the notions and properties mentioned at the beginning of this section, we give a quick proof of the lemma in question. Indeed, consider the differential equation

$$\dot{x} = X(x) = \frac{x}{t},$$

as well as its solution

$$g_t(x_0) = x_0 t.$$

The latter is defined in the neighborhood of the point x_0 , depending on how \mathcal{C}^∞ of the initial condition and is a parameter group of diffeomorphisms. We have,

$$g_0(x_0) = 0, \quad g_1(x_0) = x_0, \quad g_0^*\omega = 0, \quad g_1^*\omega = \omega.$$

Hence,

$$\begin{aligned} \omega &= g_1^*\omega - g_0^*\omega, \\ &= \int_0^1 \frac{d}{dt} g_t^* \omega dt, \\ &= \int_0^1 g_t^*(L_X \omega) dt \quad (\text{by (1)}), \\ &= \int_0^1 g_t^*(di_X \omega) dt, \quad (\text{according to (2) and the fact that } d\omega = 0) \\ &= \int_0^1 dg_t^* i_X \omega dt, \quad (\text{because } df^* \omega = f^* d\omega). \end{aligned}$$

We can therefore find a differential form λ such that $\omega = d\lambda$, where

$$\lambda = \int_0^1 g_t^* i_X \omega dt,$$

which completes the proof 2 and ends the two proofs of the lemma. \square

Remark 1 *Any exact differential form is closed. It is well known that the converse is false in general and depends on the open U on which the differential form is \mathcal{C}^1 . For example, if $U = \mathbb{R}^2 \setminus \{(0,0)\}$ then the differential form*

$$\omega = -\frac{x_2}{x_1^2 + x_2^2} dx_1 + \frac{x_1}{x_1^2 + x_2^2} dx_2,$$

is closed but is not exact. Indeed, this form is obviously closed. To show that it is not exact, we use the fact that in general if ω is an exact 1-differential form on an open subset and γ a closed path in this \mathcal{C}^1 piecewise open subset, then $\int_\gamma \omega = 0$. In the present example, $U = \mathbb{R}^2 \setminus \{(0,0)\}$ and let γ be the unit circle of parametric equations : $x_1(t) = \cos t$, $x_2(t) = \sin t$, $t \in [0, 2\pi]$. We have

$$\int_\gamma \omega = \int_0^{2\pi} \left(-\frac{x_2}{x_1^2 + x_2^2} x_1'(t) + \frac{x_1}{x_1^2 + x_2^2} x_2'(t) \right) dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi.$$

Since $\int_\gamma \omega \neq 0$, then ω is not exact. This example shows that in Poincaré lemma, the hypothesis that the open is starred is essential (here, the open $U = \mathbb{R}^2 \setminus \{(0,0)\}$ is not a star-shaped subset).

Proposition 1 *The Poincaré lemma assures the existence of λ but not its uniqueness.*

Proof : Indeed, let ω be a closed k -differential form on a star-shaped open subset of \mathbb{R}^n . By Poincaré lemma, there exist a $(k - 1)$ -differential form λ such that $\omega = d\lambda$. If μ is any $(k - 2)$ -differential form, then $\lambda + d\mu$ satisfies the same equation :

$$d(\lambda + d\mu) = d\lambda + d(d\mu) = d\lambda = \omega,$$

(because the exterior derivative obeys the rule : $d(d\mu) = 0$, see for example [7]). Conversely, if λ_1 and λ_2 are any two $(k - 2)$ -differential forms such that : $\omega = d\lambda_1 = d\lambda_2$, then $d(\lambda_1 - \lambda_2) = 0$. By Poincaré lemma, there exist a $(k - 2)$ -differential form θ such that : $\lambda_1 - \lambda_2 = d\theta$, i.e., $\lambda_1 = \lambda_2 + d\theta$. From this we deduce that the general solution can be expressed as the sum of a particular solution and the derivative of an arbitrary $(k - 2)$ -differential form. The proof is completed. \square

On manifolds the Poincaré lemma can be stated as follows :

Proposition 2 *Any closed k -differential form ω is exact in the neighborhood of an n -manifold M (or, in \mathbb{R}^n any closed differential form is exact).*

Proof : We have seen that for a star-shaped open subset of \mathbb{R}^n , the form ω is exact. Since M is locally diffeomorphic to an open subset of \mathbb{R}^n , then for every point $p \in M$ there exists also a neighborhood U of p and a $(k - 1)$ -differential form λ such that $\omega = d\lambda$ on U . \square

3 Some connections with Cech-De Rham-Dolbeault cohomologies

We will give in the subsections below several formulations of Poincaré lemma. Similarly, some examples and questions closely related to the Poincaré lemma will be discussed.

3.1 Cech-De Rham cohomologies and the Poincaré lemma

Let \mathcal{F} be a sheaf on a topological space M . Consider the set $C^k(\mathcal{U}, \mathcal{F})$ of k -cochains of degree k with values in \mathcal{F} , i.e., the set of applications that associates with each k -up of open cover of \mathcal{U} a section on their intersection. In other words, we have

$$C^k(\mathcal{U}, \mathcal{F}) = \prod_{\alpha_0, \dots, \alpha_k} \mathcal{F}(\mathcal{U}_{\alpha_0} \cap \dots \cap \mathcal{U}_{\alpha_k}),$$

where the direct product is taken over the set of all $(k+1)$ distinct elements $\alpha_0, \dots, \alpha_k$ in the index set. In particular, we have $C^0(\mathcal{U}, \mathcal{F}) = \prod_{\alpha} \mathcal{F}(\mathcal{U}_{\alpha})$ and $C^1(\mathcal{U}, \mathcal{F}) = \prod_{\alpha, \beta} \mathcal{F}(\mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta})$. We define the coboundary operator

$$\delta : C^k(\mathcal{U}, \mathcal{F}) \longrightarrow C^{k+1}(\mathcal{U}, \mathcal{F}),$$

for $s \in C^k(\mathcal{U}, \mathcal{F})$, by

$$(\delta s)(\mathcal{U}_{\alpha_0}, \dots, \mathcal{U}_{\alpha_k}) = \sum_{j=0}^{k+1} (-1)^j s(\mathcal{U}_{\alpha_0}, \dots, \widehat{\mathcal{U}_{\alpha_j}}, \dots, \mathcal{U}_{\alpha_{k+1}}) \Big|_{\mathcal{U}_{\alpha_0} \cap \dots \cap \mathcal{U}_{\alpha_{k+1}}}.$$

We easily check that $\delta^2 = 0$, so $(C(\mathcal{U}, \mathcal{F}))$ forms a cochain complex and we have

$$C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \dots$$

with

$$\begin{aligned} \delta & : C^0(\mathcal{U}, \mathcal{F}) \longrightarrow C^1(\mathcal{U}, \mathcal{F}), \\ & \quad s_{\alpha\beta} \longmapsto s_{\beta}|_{\mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta}} - s_{\alpha}|_{\mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta}}, \\ \delta & : C^1(\mathcal{U}, \mathcal{F}) \longrightarrow C^2(\mathcal{U}, \mathcal{F}), \\ & \quad s_{\alpha\beta\gamma} \longmapsto s_{\beta\gamma}|_{\mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta} \cap \mathcal{U}_{\gamma}} - s_{\alpha\gamma}|_{\mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta} \cap \mathcal{U}_{\gamma}} + s_{\alpha\beta}|_{\mathcal{U}_{\alpha} \cap \mathcal{U}_{\beta} \cap \mathcal{U}_{\gamma}}, \\ & \quad \vdots \\ \delta & : C^k(\mathcal{U}, \mathcal{F}) \longrightarrow C^{k+1}(\mathcal{U}, \mathcal{F}), \\ & \quad s_{\alpha_0 \dots \alpha_{k+1}} \longmapsto \sum_{j=0}^{k+1} (-1)^j s_{\alpha_0 \dots \widehat{\alpha_j} \dots \alpha_{k+1}} \Big|_{\mathcal{U}_{\alpha_0} \cap \dots \cap \mathcal{U}_{\alpha_{k+1}}}. \end{aligned}$$

For any sheaf \mathcal{F} of abelian groups on M and for any open cover \mathcal{U} of M , we can find an application $\mathcal{F} \longrightarrow C^0(\mathcal{U}, \mathcal{F})$, so that the sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \dots$$

be exact. The Čech cohomology of the sheaf \mathcal{F} with respect to the cover \mathcal{U} is the quotient,

$$H^k(\mathcal{U}, \mathcal{F}) = \frac{\ker [\delta : C^k(\mathcal{U}, \mathcal{F}) \longrightarrow C^{k+1}(\mathcal{U}, \mathcal{F})]}{\text{Im} [\delta : C^{k-1}(\mathcal{U}, \mathcal{F}) \longrightarrow C^k(\mathcal{U}, \mathcal{F})]}.$$

These cohomology groups depend on the cover \mathcal{U} . The Čech cohomology (or k -th sheaf cohomology) of \mathcal{F} on M is defined to be the direct limit of $H^k(\mathcal{U}, \mathcal{F})$ as \mathcal{U} becomes finer and finer,

$$H^k(M, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^k(\mathcal{U}, \mathcal{F}).$$

The direct limit means, roughly, that for something to appear in the final cohomology, it only needs to appear for sufficiently refined covers. The passage to the limit in this definition is delicate in practice although its interest is that there is no longer any dependence on covers. However, Leray theorem [5] asserts that :

Theorem 2 *If the open \mathcal{U}_α of a cover \mathcal{U} of M are such that :*

$$H^k(A, \mathcal{F}) = 0,$$

for any $k > 0$ and any finite intersection $A \equiv \mathcal{U}_{\alpha_1} \cap \dots \cap \mathcal{U}_{\alpha_k}$ of open \mathcal{U}_α , then the groups $H^k(\mathcal{U}, \mathcal{F})$ et $H^k(M, \mathcal{F})$ are isomorphic for all k .

Let Ω^k be the sheaf of k -differential forms on a manifold M and \mathcal{Z}^k be the sheaf of the closed k -forms. Let d_* be the application of cohomology associated with $d : \Omega^k \rightarrow \mathcal{Z}^{k+1}$. The k -th De Rham cohomology $H_{DR}^k(M)$ is defined as the quotient space of the k -closed differential forms by the $(k-1)$ -differential forms. Since a k -form is a global section of the sheaf of k -forms, so it is an element of $H^0(M, \Omega^k)$ because this latter group of cohomology identifies with global sections. Therefore,

$$H_{DR}^k(M) = \frac{H^0(M, \mathcal{Z}^k)}{d_* H^0(M, \Omega^{k-1})}.$$

As a consequence of the Poincaré lemma, we have the following result :

Proposition 3 *De Rham's cohomology of a manifold is isomorphic to its Čech cohomology with coefficients in \mathbb{R} .*

Proof : According to Poincaré lemma, any closed form is locally exact. So the sheaf sequence

$$0 \rightarrow \mathcal{Z}^k \rightarrow \Omega^k \xrightarrow{d} \mathcal{Z}^{k+1} \rightarrow 0,$$

is exact where $\mathcal{Z}^0 \equiv \mathbb{R}$ is the sheaf of locally constant functions and Ω^0 the sheaf of functions in \mathcal{C}^∞ . The following long exact sequence in cohomology associated with the above sequence is

$$H^q(M, \Omega^k) \xrightarrow{d^*} H^q(M, \mathcal{Z}^{k+1}) \xrightarrow{\partial} H^{q+1}(M, \mathcal{Z}^k) \rightarrow H^{q+1}(M, \Omega^k),$$

where d^* denotes the exterior derivative and ∂ the boundary operator of the long exact sequence. The sheaf Ω^p admits partitions of the unit, hence for $q \geq 1$,

$$H^q(M, \Omega^k) = 0,$$

and so

$$H^q(M, \mathcal{Z}^{k+1}) = H^{q+1}(M, \mathcal{Z}^k).$$

For the particular case $q = 0$, we have

$$H^0(M, \Omega^k) \xrightarrow{d^*} H^0(M, \mathcal{Z}^{k+1}) \xrightarrow{\partial} H^1(M, \mathcal{Z}^p) \longrightarrow 0,$$

and

$$H^1(M, \Omega^k) = \frac{H^0(M, \mathcal{Z}^{k+1})}{d_* H^0(M, \Omega^k)}.$$

Therefore,

$$H_{DR}^k(M) = H^1(M, \mathcal{Z}^{k-1}) = H^{k-1}(M, \mathcal{Z}^1) = H^k(M, \mathcal{Z}^0) = H^k(M, \mathbb{R}),$$

and the result follows. \square

Let M and N two manifolds. Recall that two maps $f : M \rightarrow N$ and $g : N \rightarrow M$ are (smoothly) homotopic, if there exists a (smooth) homotopy

$$h : M \times I \rightarrow N, \quad h(x, 0) = f(x), \quad h(x, 1) = g(x),$$

(I is the interval, $0 \leq t \leq 1$). We say the manifolds M and N are homotopy equivalent if there exist (smooth) maps $f : M \rightarrow N$ and $g : N \rightarrow M$ such that the composites $gof : M \rightarrow M$ and $fog : N \rightarrow N$ are (smoothly) homotopic to the respective identity maps $M \rightarrow M$, $x \mapsto x$ and $N \rightarrow N$, $y \mapsto y$. For example, the closed unit disc in \mathbb{R}^n is homotopy equivalent to a point. It easy to verify that $\mathbb{R}^n \setminus \{0\}$ and S^{n-1} are homotopy equivalent. Using reasoning similar to proof 2, we show that :

Proposition 4 *If two manifolds M and N are homotopy equivalent then their cohomology groups are isomorphic.*

Proof : Indeed, consider the (smooth) homotopy $F : M \times I \rightarrow N$ between the above two maps f and g . Then the induced homomorphisms

$$\begin{aligned} f^* & : H^k(N) \rightarrow H^k(M), \\ g^* & : H^k(M) \rightarrow H^k(N), \end{aligned}$$

of the cohomology groups coincide. It suffices to consider a differential form $\omega \in H^k(gof)(M)$, which implies that it exists a pullback

$$\begin{aligned} h^*(\omega) & = \lambda + dt \wedge \theta, \\ h^*(\omega)|_{t=t_0} & = h^*(\cdot, t_0)(\omega) = \lambda|_{t=t_0}, \end{aligned}$$

where $\lambda \in H^k(M \times I)$, $\theta \in H^{k-1}(M \times I)$, and which λ, θ do not involve the differential dt (in the sens that λ, θ do not contain a dt term). We have¹

$$h^*(d_M(\omega)) = d_{M \times I}(h^*(\omega)) = dt \wedge \left(\frac{\partial \lambda}{\partial t} - d_M(\theta) \right),$$

1. Let $V \subset \mathbb{R}^m$, $U \subset \mathbb{R}^n$ be open subsets, $g : V \rightarrow U$ be a differentiable application and $\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$, a k -differential form on U . If ω is \mathcal{C}^1 on U and g is \mathcal{C}^2 on V , then (see for example [8]), $g^*(d\omega) = d(g^*\omega)$.

and since ω is closed then $h^*(d_M(\omega)) = 0$, so

$$\frac{\partial \lambda}{\partial t} = d_M(\theta).$$

Moreover, we have

$$\begin{aligned} h^*(., 1)(\omega) - h^*(., 0)(\omega) &= \lambda|_{t=1} - \lambda|_{t=0}, \\ &= \int_0^1 \frac{\partial \lambda}{\partial t} dt, \\ &= \int_0^1 d_M(\theta) dt, \\ &= d_M \left(\int_0^1 \theta dt \right). \end{aligned}$$

Therefore,

$$(gof)^* = h^*(., 1),$$

is the identity map hence, f^* and g^* are inverse to each other and in particular, both are isomorphisms. More precisely, since $gof \sim id_M$ and $fog \sim id_N$, we deduce that

$$\begin{aligned} f^*og^* &= (gof)^* = id_M^* = id_{H^k(M)}, \\ g^*of^* &= (fog)^* = id_N^* = id_{H^k(N)}. \end{aligned}$$

Hence, f^* is a vector space isomorphism and $(f^*)^{-1} = g^*$, which completes the proof. \square

Remark 2 Recall that a (smooth) manifold M is contractible if the identity map on M is homotopic to a constant map or in other words if M is homotopy equivalent to a point. For example, a star-shaped open subset of \mathbb{R}^n is contractible and hence has De Rham cohomology of a point. The cohomology groups of space \mathbb{R}^n (and of the ball around any point p) are isomorphic to those of p . Thus $H^k(\mathbb{R}^n)$ is trivial for $k > 0$, while $H^0(\mathbb{R}^n) \simeq \mathbb{R}$. In fact if M is contractible, which means that M is homotopy equivalent to a point and since homotopy equivalent manifolds admit isomorphic De Rham cohomology groups, then

$$H^k(M) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k > 0 \end{cases}$$

This fact leads immediately to the Poincaré Lemma. As we mentioned above when $k = 0$, notice that constant functions are closed and $H^0(M)$ is a finite dimensional vector space equal to the number of connected components of M . Here we have only 0-differential forms, i.e., $f(x)$ functions on M . There are no exact differential forms. Then, $H^0(M, \mathbb{R}) = \{f : f \text{ is closed}\}$. Since $df(x) = 0$, then in any chart (U, x_1, \dots, x_n) of M , we have

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \dots = \frac{\partial f}{\partial x_n} = 0.$$

Therefore, $f(x) = \text{constant}$ locally, i.e., $f(x) = \text{constant}$ on each connected component of M . So the number of connected components of M is the dimension in question.

Remark 3 As we have seen the Poincaré lemma and De Rham theorem give a characterization of the de Rham cohomology groups. Applying Poincaré lemma for k -differential forms where $k \neq 0$, we show that the cohomology of a ball is trivial and in particular, $\dim H^k(\mathbb{R}^n) = \delta(k, 0)$. If S^n is the n -sphere and if $T^n = (S^1)^n$ is the n -dimensional torus, i.e., the product of n circles, then by proceeding by induction on n , we obtain

$$\dim H^k(S^n) = \delta(n, k), \quad \dim H^k(T^n) = \frac{n!}{k!(n-k)!}.$$

3.2 Dolbeault cohomology : $\bar{\partial}$ -Poincaré lemma (or Dolbeault-Grothendieck lemma)

In each point of a complex manifold M , we define local coordinates z_j and \bar{z}_j as well as the tangent bundle $\mathbb{C} \left[\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \right]$ that we note TM . This admits the decomposition $TM = T'M \oplus T''M$, where $T'M$ (resp. $T''M$) is the holomorphic (respectively antiholomorphic) part of TM generated by $\frac{\partial}{\partial z_j}$ (resp. $\frac{\partial}{\partial \bar{z}_j}$). Similarly, the cotangent bundle T^*M (dual of TM) admits the decomposition : $T^*M = T'^*M \oplus T''^*M$, into holomorphic and antiholomorphic parts. The points of this bundle are the linear forms on the fibers of $TM \rightarrow M$. A p -differential form is a section of the bundle $\Lambda^p T^*M$ (the exterior powers of T^*M). The points of the latter being the alternating multilinear p -forms on T^*M . If z_1, \dots, z_n are holomorphic local coordinates on a complex manifold M of dimension n , then $dz_1, \dots, dz_n, d\bar{z}_1, \dots, d\bar{z}_n$ form a local base of the space of differential forms. A differential form on M of type (p, q) is given locally by

$$\omega = \sum_{\substack{1 \leq j_1 < \dots < j_p \leq n \\ 1 \leq k_1 < \dots < k_q \leq n}} f_{j_1 \dots j_p k_1 \dots k_q} dz_{j_1} \wedge \dots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \dots \wedge d\bar{z}_{k_q},$$

where $f_{j_1 \dots j_p k_1 \dots k_q}$ are \mathcal{C}^∞ functions with complex values. This definition does not depend on the choice of holomorphic local coordinates. Obviously, a form of type $(0, 0)$ is a function. The set of differential forms of type (p, q) over M is a complex vector bundle denoted $A_M^{p,q}$. It should be noted that only $A_M^{p,0}$

is holomorphic. The differential of ω is

$$\begin{aligned}
d\omega &= \sum_{\substack{1 \leq j_1 < \dots < j_p \leq n \\ 1 \leq k_1 < \dots < k_q \leq n}} df_{j_1 \dots j_p k_1 \dots k_q} \wedge dz_{j_1} \wedge \dots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \dots \wedge d\bar{z}_{k_q}, \\
&= \sum_{\substack{1 \leq j_1 < \dots < j_p \leq n \\ 1 \leq k_1 < \dots < k_q \leq n}} \partial f_{j_1 \dots j_p k_1 \dots k_q} \wedge dz_{j_1} \wedge \dots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \dots \wedge d\bar{z}_{k_q} \\
&\quad + \sum_{\substack{1 \leq j_1 < \dots < j_p \leq n \\ 1 \leq k_1 < \dots < k_q \leq n}} \bar{\partial} f_{j_1 \dots j_p k_1 \dots k_q} \wedge dz_{j_1} \wedge \dots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \dots \wedge d\bar{z}_{k_q}, \\
&= \partial\omega + \bar{\partial}\omega,
\end{aligned}$$

where $\partial \equiv \frac{\partial}{\partial z} dz$ and $\bar{\partial} \equiv \frac{\partial}{\partial \bar{z}} d\bar{z}$. Since $\partial f_{j_1 \dots j_p k_1 \dots k_q}$ is of type $(1, 0)$ and $\bar{\partial} f_{j_1 \dots j_p k_1 \dots k_q}$ is of type $(0, 1)$, then the component $\partial\omega$ of $d\omega$ is of type $(p+1, q)$ and $\bar{\partial}\omega$ is of type $(p, q+1)$. So we have the following decomposition : $d = \partial + \bar{\partial}$ and the operators $\partial, \bar{\partial}$ are independent of the choice of holomorphic local coordinates. The relation $d^2 = 0$ means here that :

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Indeed, we deduce from the relations $d^2 = 0$ and $d = \partial + \bar{\partial}$, that

$$0 = d^2 = \partial^2 + \bar{\partial}^2 + \partial\bar{\partial} + \bar{\partial}\partial.$$

Note that if ω is a form of type (p, q) , then $\partial^2\omega$ is of type $(p+2, q)$, $\bar{\partial}^2\omega$ is of type $(p, q+2)$ and $(\partial\bar{\partial} + \bar{\partial}\partial)$ is of type $(p+1, q+1)$. The result is that the sum of these three forms is zero. The operator $\bar{\partial}$ satisfies the following rule (Leibniz rule) :

$$\bar{\partial}(\omega \wedge \lambda) = \bar{\partial}\omega \wedge \lambda + (-1)^{\deg\omega} \omega \wedge \bar{\partial}\lambda.$$

Noticing that $\partial\omega = \overline{\bar{\partial}\omega}$, we also obtain

$$\partial(\omega \wedge \lambda) = \partial\omega \wedge \lambda + (-1)^{\deg\omega} \omega \wedge \partial\lambda.$$

Moreover, the differential form ω is said $\bar{\partial}$ -closed if $\bar{\partial}\omega = 0$ and $\bar{\partial}$ -exact if there exists a form λ such that : $\omega = \bar{\partial}\lambda$. Any $\bar{\partial}$ -exact form is $\bar{\partial}$ -closed. The analogue of Poincaré lemma for the Dolbeault operator $\bar{\partial}$ (see below for definition) is the $\bar{\partial}$ -Poincaré lemma (or Dolbeault-Grothendieck lemma) and can be stated as follows :

Proposition 5 *If ω is a \mathcal{C}^1 form of type (p, q) , $q > 0$ with $\bar{\partial}\omega = 0$, then it exists locally on M a \mathcal{C}^1 form λ of type $(p, q-1)$ such that : $\omega = \bar{\partial}\lambda$.*

We have shown previously that Poincaré lemma leads to the nullity of cohomology groups in the case of star-shaped open subsets. We have an isomorphism

$$H_{\bar{\partial}}^{p,q}(M) \simeq H^q(M, \Omega_M^p),$$

for all integers p and q , where

$$H_{\bar{\partial}}^{p,q}(M) = \frac{\{\bar{\partial}\text{-closed forms of type } (p, q) \text{ on } M\}}{\{\bar{\partial}\text{-exact forms of type } (p, q) \text{ on } M\}},$$

is a Dolbeault cohomology group of complex manifold M .

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